Convergent Sequence

Definition

A sequence $\{p_n\}$ in a metric space (X, d) is said to *converge* if there is a point $p \in X$ with the following property:

$$(\forall \epsilon > 0)(\exists N)(\forall n > N) d(p_n, p) < \epsilon$$

In this case we also say that $\{p_n\}$ converges to p or that p is the limit of $\{p_n\}$ and we write $p_n \to p$ or $\lim_{n\to\infty} p_n = p$.

If $\{p_n\}$ does not converge we say it *diverges*

If there is any ambiguity we say $\{p_n\}$ *converges/diverges in X*

The set of all p_n is said to be the *range* of $\{p_n\}$ (which may be infinite or finite). We say $\{p_n\}$ is *bounded* if the range is bounded.

Examples

Notice that our definition of convergent depends not only on $\{p_n\}$ but also on X.

For example $\{1/n : n \in \mathbb{N}\}$ converges in $\mathbb{R}^1$ and diverges in $(0, \infty)$.

consider the following sequence of complex number (i.e. $X = R^2$)

(a) If $s_n = 1/n$ then $\lim_{n\to\infty} s_n = 0$; the range is infinite, and the sequence is bounded.

(b) If $s_n = n^2$ then the sequence $\{s_n\}$ is divergent; the range is infinite, and the sequence is unbounded.

(c) If $s_n = 1 + [(-1)^n/n]$ then the sequence $\{s_n\}$ converges to 1, is bounded, and has infinite range.

(d) If $s_n = i^n$ the sequence $\{s_n\}$ is divergent, is bounded and has finite range.

(e) If $s_n = 1$ $(n = 1, 2, 3, \dots)$ then $\{s_n\}$ converges to 1is bounded

Properties of Convergent Sequences

Theorem

(a) $\{p_n\}$ *converges to* $p \in X$ *if and only if every neighborhood of* p *contains* p_n *for all but finitely many* n.

(b) *If* $p, p' \in X$ *and if* $\{p_n\}$ *converges to* p *and to* p' *then* $p = p'$

(c) *If* $\{p_n\}$ *converges then* $\{p_n\}$ *is bounded.*

(d) *If* $E \subseteq X$ *and if* p *is a limit point of* E, *then there is a sequence* $\{p_n\}$ *in* E *such that* $p = \lim_{n\to\infty} p_n$

Properties of Convergent Sequences

Theorem

Suppose $\{s_n\}, \{t_n\}$ *are complex sequence with* $\lim_{n\to\infty} s_n = s$ *and* $\lim_{n\to\infty} t_n = t$. *Then*

(a) $\lim_{n\to\infty}(s_n + t_n) = s + t$

(b) $\lim_{n\to\infty} c \cdot s_n = c \cdot s$ *and* $\lim_{n\to\infty} c + s_n = c + s$ *for any number* c.

(c) $\lim_{n\to\infty} s_n t_n = st$

(d) $\lim_{n\to\infty} \frac{1}{s_n} = \frac{1}{s}$

Properties of Convergent Sequences

Theorem

(a) *Suppose* $\mathbf{x}_n \in \mathbb{R}^k (n \in \mathbb{N})$ *and* $\mathbf{x}_n = (\alpha_{1,n}, \ldots, \alpha_{k,n})$. *Then* $\{\mathbf{x}_n\}$ *converges to* $\mathbf{x} = (\alpha_1, \ldots, \alpha_k)$ *if and only if*

$$\lim_{n\to\infty} \alpha_{j,n} = \alpha_j \ (1 \leq j \leq k)$$

(b) *Suppose* $\{\mathbf{x}_n\}, \{\mathbf{y}_n\}$ *are sequences in* $\mathbb{R}^k$, $\{\beta_n\}$ *is a sequence of real numbers, and* $\mathbf{x}_n \to \mathbf{x}, \mathbf{y}_n \to y, \beta_n \to \beta$. *Then*

$$\lim_{n\to\infty} (\mathbf{x}_n + \mathbf{y}_n) = \mathbf{x} + \mathbf{y}$$

$$\lim_{n\to\infty} (\mathbf{x}_n \cdot \mathbf{y}_n) = \mathbf{x} \cdot \mathbf{y}$$

$$\lim_{n\to\infty} \beta_n \mathbf{x}_n = \beta \mathbf{x}$$

Subsequences

Definition

Given a sequence $\{p_n\}$, consider a sequence $\{n_k\}$ of positive integers such that $n_1 < n_2 < n_3 < \ldots$. Then the sequence $\{p_{n_i}\}$ is called a *subsequence* of $\{p_n\}$. If $\{p_{n_i}\}$ converges its limit is called a *subsequential limit* of $\{p_n\}$.

It is clear that $\{p_n\}$ converges to p if and only if every subsequence of $\{p_n\}$ converges to p.

Subsequences and Compact Metric Spaces

Theorem

(a) *If* $\{p_n\}$ *is a sequence in a compact metric space* X*, then some subsequence of* $\{p_n\}$ *converges to a point of* X*.*

(b) *Every bounded sequence in* $\mathbb{R}^k$ *contains a convergent subsequence.*

Subsequences Limits

Theorem

The subsequential limits of a sequence $\{p_n\}$ in a metric space X form a closed subset of X.

Cauchy Sequence

Definition

A sequence $\{p_n\}$ in a metric space (X, d) is said to be a *Cauchy sequence* if for every $\epsilon > 0$ there is an integer N such that $d(p_n, p_m) < \epsilon$ for all $n, m \geq N$.

Definition

Let E be a nonempty subset of a metric space (X, d), and let $S = \{d(p, q) : p, q \in E\}$. The *diameter of E* is sup S.

If $\{p_n\}$ is a sequence in X and if E_N consists of the points $p_N, p_{N+1}, \ldots,$ it is clear that $\{p_n\}$ is a *Cauchy sequence* if and only if

$$\lim_{N\to\infty} diam E_N = 0$$

Cauchy Sequences and Closed Sets

Theorem

(a) *If $\overline{E}$ is the closure of a set E in a metric space X, then*

$$\text{diam } \overline{E} = \text{diam } E$$

(b) *If K_n is a sequence of compact sets in X such that $K_n \subset K_{n+1}$ $(n \in \mathbb{N})$ and if $\lim_{n\to\infty} \text{diam } K_n = 0$ then $\bigcap_1^\infty K_n$ consists of exactly one point.*

Cauchy Sequences and Convergent Sequences

Theorem

(a) *In any metric space X, every convergent sequence is a Cauchy sequence.*

(b) *If X is a compact metric space and if $\{p_n\}$ is a Cauchy sequence in X then $\{p_n\}$ converges to some point of X.*

(c) *In $\mathbb{R}^k$ every Cauchy sequence converges.*

Complete Spaces

Definition

A metric space is said to be *complete* if every Cauchy sequence converges.

Notice that all compact metric spaces are complete but there are metric spaces (like $\mathbb{R}^k$) which are complete but not compact.

Lemma

Every closed subset of a complete metric space is complete.

Increasing/Decreasing Sequences

Definition

A sequence $\{s_n\}$ of real numbers is said to be

(a) *monotonically increasing* if $s_n \leq s_{n+1}$ for all $n \in \mathbb{N}$

(b) *monotonically decreasing* if $s_n \geq s_{n+1}$ for all $n \in \mathbb{N}$

(c) *monotonic* if it is monotonically increasing or monotonically decreasing.

Theorem

Suppose $\{s_n\}$ is monotonic. Then $\{s_n\}$ converges if and only if $\{s_n\}$ is bounded.