Definition of Metric Spaces

Definition

A pair (X, d) is a called a *metric space* if

- X is a set, whose elements we shall call *points*.
- $d : X \times X \to \mathbb{R}$ is a function called the *distance function*.
- For any two points $p, q \in X$.
  (a) $d(p, q) > 0$ if $p \neq q$ and $d(p, p) = 0$.
  (b) $d(p, q) = d(q, p)$
  (c) $d(p, q) \leq d(p, q) + d(q, r)$ for any $r \in X$

Condition (c) is called the *triangle inequality*.

Theorem

Notice that if (X, d) is a metric space and $Y \subseteq X$ then so is (Y, d).

$\mathbb{R}^k$ as a Metric Spaces

The most important example of metric spaces, for us, are Euclidean Spaces.

Definition

For all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^k$ let $d(\mathbf{x}, \mathbf{y}) = |\mathbf{x} - \mathbf{y}|$. Then $(\mathbb{R}^k, d)$ is a metric space.

Convex Sets

Definition

The *segment* (a, b) is the set $\{x : a < x < b\}$. The *interval* $[a, b] = \{x : a \leq x \leq b\}$. The *half open interval* $[a, b) = \{x : a \leq x < b\}$ and $(a, b] = \{x : a < x \leq b\}$.

Definition

If $a_i < b_i$ for $i = 1 \ldots k$, then $\{(x_1, \ldots, x_k) : a_i \leq x_i \leq b_i\} \subseteq \mathbb{R}^k$ is called a k-cell.

So a 1-cell is an interval, a 2-cell is a rectangle, etc.

Balls

Definition

If (X, d) is a metric space, $x \in X$ and $r > 0$ is a real then

- $B(x, r) = \{y \in X : d(x, y) < r\}$ is the *open ball of radius r at x*
- $\overline{B}(x, r) = \{y \in X : d(x, y) \leq r\}$ is the *closed ball of radius r at x*

Convex Sets

Definition

We say a set $E \subseteq \mathbb{R}^k$ is *covex* if for all $\mathbf{x}, \mathbf{y} \in E$ and all $0 < \gamma < 1$

$$\gamma \mathbf{x} + (1 - \gamma)\mathbf{y} \in E$$

Theorem

Any ball (open or closed) in $\mathbb{R}^k$ is convex.

Theorem

All k-cells are convex.

Definitions For a Metric Space

Definition

Let (X, d) be a metric space. All points and sets mentioned are elements or subsets of X.

(a) A *neighborhood* of p is a set $N_r(p)$ consisting of all q such that $d(p, q) < r$ for some $r > 0$.

(b) A point p is a *limit point* of the set E if *every* neighborhood of p contains a point $q \neq p$ such that $q \in E$

(c) If $p \in E$ and p is not a limit point of E then p is called an *isolated point* of E.

(d) E is *closed* if every limit point of E is a point of E.

Definitions For a Metric Space

Definition

(e) A point p is an *interior point* of E if there is a neighborhood N of p such that $N \subseteq E$.

(f) E is *open* if every point of E is an interior point of E.

(g) The *complement* of E (denoted by E^c) is the set of all points $p \in X$ such that $p \neq E$.

(h) E is *perfect* if E is closed and if every point of E is a limit point of E.

Definitions For a Metric Space

Definition

(i) E is *bounded* if there is a real number M and a point $q \in X$ such that $d(p,q) < M$ for all $p \in E$.

(j) E is *dense in* X if every point of X is a limit point of E or a point of E (or both).

Note that in $\mathbb{R}^1$ neighborhoods are segments, in $\mathbb{R}^2$ neighborhoods are interiors of circles, and in $\mathbb{R}^k$ neighborhoods are interiors of k-spheres.

Open Neighborhoods

Theorem

Every neighborhood is open

Limit Points

Theorem

If p is a limit point of a set E, then every neighborhood of p contains infinitely many points of E.

Corollary

A finite set has no limit points

Examples of Closed Sets

Closed Sets:

- $\mathbb{R}^2$. This is also a perfect set.
- The set of all complex numbers z with $|z| \leq 1$. This is also perfect and bounded.
- Any non-empty finite set. This is also bounded.
- The set of integers.

Note the set $E = \{1/n : n \in \mathbb{N}\}$ is not closed (nor perfect) because while it has a limit point ($z = 0$) no point of E is a limit point of E.

Examples of Open Sets

Open Sets:

- $\mathbb{R}^2$. This is also a perfect set.
- The set of all complex numbers z with $|z| < 1$. This is bounded.

Note that the interval (a, b), while open in $\mathbb{R}^1$ is not open when considered as a subset of $\mathbb{R}^2$.

Complements of Unions

Theorem

Let $\{E_\alpha : \alpha \in I\}$ *be a (finite or infinite) collection of sets. Then*

$$\left(\bigcup_{\alpha \in I} E_\alpha\right)^c = \bigcap_{\alpha \in I} E_\alpha^c$$

Opens and Closed Sets

Theorem

A set E is open if and only if its complement is closed.

Corollary

A set F is closed if and only if its complement is open.

Opens and Closed Sets

Theorem

(a) *For any collections* $\{G_\alpha : \alpha \in I\}$ *of open sets* $\bigcup_{\alpha \in I} G_\alpha$ *is open.*

(b) *For any collections* $\{F_\alpha : \alpha \in I\}$ *of closed sets* $\bigcap_{\alpha \in I} F_\alpha$ *is closed.*

(c) *For any finite collection* $G_1, \ldots, G_n$ *of open sets* $\bigcap_{i=1}^{n} G_n$ *is open.*

(d) *For any finite collection* $F_1, \ldots, F_n$ *of closed sets* $\bigcup_{i=1}^{n} F_n$ *is closed.*

Let $E_n = (-1/n, 1/n)$. Then $\bigcap_{n \in \mathbb{N}} E_n = \{0\}$ which isn't open. So the intersection of infinitely many open sets may not be open.

Closure

Definition

If (X, d) is a metric space and $E \subseteq X$ then let E' be the set of limit points of E in X. The *closure* of E is $\overline{E} = E \cup E'$.

Theorem

If (X, d) is a metric space and $E \subseteq X$ then

(a) $\overline{E}$ *is closed*

(b) $E = \overline{E}$ *if and only if E is closed.*

(c) $\overline{E} \subseteq F$ *for every closed set $F \subseteq X$ such that $E \subseteq F$.*

Bounded Closed Sets

Theorem

Let E be a non-empty set of real numbers which is bounded above. Let $y = supE$. Then $y \in \overline{E}$. Hence $y \in E$ if E is closed.

Relative Closure

Definition

Suppose (X, d) is a metric space with $E \subset Y \subset X$. We say that E is *open relative to* Y if for all $p \in E$ there is an $r_p > 0$ such that for all $q \in Y$ if $d(p, q) < r_p$ then $q \in E$.

So E is open relative to Y if and only if E is open in the metric space (Y, d)

Theorem

Suppose (X, d) is a metric space and $Y \subseteq X$. A subset E of Y is open relative to Y if and only if $E = Y \cap G$ for some open subset G of X