

# Complex Numbers

## Definition

A *complex number* is an ordered pair  $(a, b)$  where  $a$  and  $b$  are real numbers.

Here an ordered pair means that  $(a, b) \neq (b, a)$  if  $a \neq b$ .

## Definition

Suppose  $x = (a, b)$  and  $y = (c, d)$  are complex numbers. Then we say  $x = y$  if and only if  $a = c$  and  $b = d$

We define

$$x + y = (a + c, b + d)$$

$$x \cdot y = (ac - bd, ad + bc)$$

# Complex Numbers Are A Field

## Theorem

*The complex numbers with this addition and multiplication are a field.*

# Complex Numbers Are A Field

## Theorem

*For any real numbers  $a$  and  $b$  we have*

$$(a, 0) + (b, 0) = (a + b, 0)$$

$$(a, 0) \cdot (b, 0) = (a \cdot b, 0)$$

In particular this means that if we identify  $(a, 0)$  with  $a$  then the field of real numbers is a subfield of the complex numbers.

### Definition

Let  $i = (0, 1)$

### Theorem

$$i^2 = -1$$

### Theorem

*If  $a$  and  $b$  are real numbers then  $(a, b) = a + b \cdot i$*

# Complex Conjugate

## Definition

If  $a$  and  $b$  are real numbers and  $z = a + bi$  then the complex number  $\bar{z} = a - bi$  is called the *complex conjugate* of  $z$ .

The numbers  $a$  and  $b$  are also called the *real part* and the *imaginary part* of  $z$  respectively. We sometimes write

$$a = \operatorname{Re}(z), \quad b = \operatorname{Im}(z)$$

# Complex Conjugate Results

## Theorem

(a)  $\overline{z + w} = \bar{z} + \bar{w}$

(b)  $\overline{z \cdot w} = \bar{z} \cdot \bar{w}$

(c)  $z + \bar{z} = 2\operatorname{Re}(z)$  and  $z - \bar{z} = 2i\operatorname{Im}(z)$

(d)  $z \cdot \bar{z}$  is real and positive unless  $z = 0$ .

# Absolute Value

## Definition

If  $z$  is a complex number its *absolute value*,  $|z|$  is the non-negative square root of  $z\bar{z}$  ( $|z| = (z\bar{z})^{1/2}$ )

Note that the absolute value is well define because  $z\bar{z}$  is always non-negative and we have shown that there is always a unique positive square root of any non-negative number.

## Theorem

If  $x$  is real then  $\bar{x} = x$  and  $|x| = \sqrt{x^2}$ . So  $|x| = x$  if  $x \geq 0$  and  $|x| = -x$  if  $x < 0$ .

# Absolute Value Results

## Theorem

- (a)  $|z| > 0$  unless  $z = 0$ .  $|0| = 0$
- (b)  $|z| = |\bar{z}|$
- (c)  $|zw| = |z||w|$
- (d)  $|\operatorname{Re}(z)| \leq |z|$
- (e)  $|z + w| \leq |z| + |w|$

# Schwarz Inequality

## Definition

If  $x_1, x_2, \dots, x_n$  are complex numbers we write

$$x_1 + x_2 + \cdots + x_n = \sum_{j=1}^n x_j$$

## Theorem (Schwarz Inequality)

If  $a_1, \dots, a_n$  and  $b_1, \dots, b_n$  are complex numbers then

$$\left| \sum_{j=1}^n a_j \bar{b}_j \right|^2 \leq \sum_{j=1}^n |a_j|^2 \sum_{j=1}^n |\bar{b}_j|^2$$

## Misc. Theorem

## Theorem

Let

$$a_0z^n + a_1z^{n-1} + \dots + a_n = 0$$

*be a polynomial equation with real coefficients. If the complex number  $z = a + bi$  is a root of this equation then so is its complex conjugate  $\bar{z} = a - bi$*