Limit Points of Sequences of Functions

Theorem (7.11)

Suppose $f_n \to f$ uniformly on a set E in a metric space. Let x be a limit point of E, and suppose that

$$\lim_{t \to x} f_n(t) = A_n \quad (n = 1, 2, 3, \dots)$$

Then $\{A_n\}$ converges and

$$\lim_{t \to x} f(t) = \lim_{n \to \infty} A_n$$

Theorem (7.12)

If $\{f_n\}$ is a sequence of continuous functions on E, and if $f_n \to f$ uniformly on E, then f is continuous on E.

Limit Points of Sequences of Functions

Theorem (7.13)

Suppose K is compact and

(a) $\{f_n\}$ *is a sequence of continuous functions on K*

(b) $\{f_n\}$ *converges pointwise to a continuous function f on K*

(c) $f_n(x) \geq f_{n+1}(x)$ *for all* $x \in K$, $n = 1, 2, 3, \ldots$

Then $f_n \to f$ *uniformly on K*

Space of Continuous Functions

Definition (7.14)

If X is a metric space, $\mathcal{C}(X)$ will denote the set of all complex-valued, continuous, bounded functions with domain X.

Note that boundedness is redundant if X is compact.

Space of Continuous Functions

Definition (7.14)

We associate with each $f \in \mathcal{C}(X)$ its *supremum norm*

$$\|f\| = \sup_{x \in X} |f(x)|$$

Since f is assumed to be bounded $\|f\| < \infty$.

It is obvious that $\|f\| = 0$ only if $f(x) = 0$ for every $x \in X$ that is only if $f = 0$. If $h = f + g$, then

$$|h(x)| \le |f(x)| + |g(x)| \le \|f\| + \|g\|$$

for all $x \in X$; hence

$$\|f + g\| \le \|f\| + \|g\|$$

Space of Continuous Functions

Definition (7.14)

If we define the distance between $f, g \in \mathcal{C}(X)$ as $\|f - g\|$ then this makes $\mathcal{C}(X)$ into a metric space.

The previous theorem can be restated as saying

Theorem

A sequence $\{f_n\}$ converges to f with respect to the metric on $\mathcal{C}(X)$ if and only if $f_n \to f$ uniformly on X.

Accordingly closed subsets of $\mathcal{C}(X)$ are sometimes called *uniformly closed* and the closure of $A \subseteq \mathcal{C}(X)$ is called its *uniform closure*

Theorem (7.15)

$\mathcal{C}(X)$ is a complete metric space.

Integration of Sequences of Functions

Theorem (7.16)

Let α be monotonically increasing on $[a, b]$. Suppose $f_n \in \mathcal{R}(\alpha)$ on $[a, b]$ for $n = 1, 2, 3, \ldots$ and suppose $f_n \to f$ uniformly on $[a, b]$. Then $f \in \mathcal{R}(\alpha)$ on $[a, b]$ and

$$\int_a^b f \, d\alpha = \lim_{n \to \infty} \int_a^b f_n \, d\alpha$$

(The existence of the limit is part of the conclusion).

Integration of Sequences of Functions

Corollary

If $f_n \in \mathcal{R}(\alpha)$ on $[a, b]$ and if

$$f(x) = \sum_{n=1}^{\infty} f_n(x) \quad (a \le x \le b)$$

with the series converging uniformly on $[a, b]$ then

$$\int_a^b f \, d\alpha = \sum_{n=1}^{\infty} \int_a^b f_n \, d\alpha$$

Differentiation of Sequences of Functions

Theorem (7.17)

Suppose $\{f_n\}$ is a sequence of functions each differentiable on $[a, b]$ and such that $\{f_n(x_0)\}$ converges for some point x_0 on $[a, b]$. If $\{f_n'\}$ converges uniformly on $[a, b]$ then $\{f_n\}$ converges uniformly on $[a, b]$ to a function f, and

$$f'(x) = \lim_{n \to \infty} f_n'(x) \quad (a \leq x \leq b)$$

Nowhere Differentiable Function

Theorem (7.18)

There exists a real continuous function on the real line which is nowhere differentiable.