Limits of a Function

Definition (7.1)

Suppose $\{f_n\}$, $n = 1, 2, 3, \ldots$ is a sequence of functions defined on a set E, and suppose that the sequence of numbers $\{f_n(x)\}$ converges for every $x \in E$. We can then define a function f by

$$f(x) = \lim_{n \to \infty} f_n(x) \quad (x \in E)$$

We say that $\{f_n\}$ converges on E and that f is the *limit*,or the *limit function* of $\{f_n\}$. Sometimes we will also say "$\{f_n\}$ converges to f *pointwise* on E"

Limits of a Function

Definition

If $\sum f_n(x)$ converges for every $x \in E$ we define

$$f(x) = \sum_{n=1}^{\infty} f_n(x) \quad (x \in E)$$

We say f is the *sum* of the series $\sum f_n(x)$

Main Question

The main question we are asking is "what properties of functions are preserved when we take limits"

For example, to say f is continuous at x is to say

$$\lim_{t \to x} f(t) = f(x)$$

So to ask if the limit of a sequence of functions is continuous at x is to ask if

$$\lim_{t \to x} \lim_{n \to \infty} f_n(t) = \lim_{n \to \infty} \lim_{t \to x} f_n(t)$$

Example

Example

For $m = 1, 2, \ldots, n = 1, 2, \ldots$ let

$$s_{n,m} = \frac{m}{n+m}$$

Then for every fixed n

$$\lim_{m\to\infty} s_{m,n} = 1 \text{ so } \lim_{n\to\infty} \lim_{m\to\infty} s_{m,n} = 1$$

But for every fixed m

$$\lim_{n\to\infty} s_{m,n} = 0 \text{ so } \lim_{m\to\infty} \lim_{n\to\infty} s_{m,n} = 0$$

Example

Example

Let

$$f_n(x) = \frac{x^2}{(1+x^2)^n} \quad (x \in \mathbb{R}; n = 0, 1, \dots)$$

and consider

$$f(x) = \sum_{n=0}^{\infty} f_n(x) = \sum_{n=0}^{\infty} \frac{x^2}{(1+x^2)^n}$$

So as $f_n(0) = 0$ for all n we have $f(0) = 0$. But if $x \neq 0$ then the sum is a geometric series with sum $1 + x^2$.

So the convergent series of continuous functions may have a discontinuous sum.

Example

Example

Let

$$f_n(x) = \frac{\sin(nx)}{\sqrt{n}} \quad (x \in \mathbb{R}; n = 1, 2, \dots)$$

and

$$f(x) = \lim_{n\to\infty} f_n(x) = 0$$

then $f'(x) = 0$ for all x and

$$f_n'(x) = \sqrt{n}\cos(nx)$$

So $\{f_n'\}$ does not converge to f'. For instance

$$f_n'(0) = \sqrt{n} \to \infty$$

but $f'(0) = 0$.

Example

Example

Let

$$f_n(x) = n^2 x(1 - x^2)^n \qquad (x \in [0, 1]; n = 1, 2, \dots)$$

then we have $f_n(0) = 0$ for all n and if $0 < x \leq 1$ we have

$$\lim_{n \to \infty} f_n(x) = 0$$

So if we let $f(x) = \lim_{n \to \infty} f_n(x)$ for $x \in [0, 1]$ then $f(x)$ is the constant 0 and

$$\int_0^1 f(x) dx = 0$$

Example

Example

But we also have

$$\int_0^1 f_n(x)dx = \int_0^1 n^2x(1-x^2)^n dx = \frac{n^2}{2n+2}$$

Hence $\int_0^1 f_n(x)dx \to \infty$ as $n \to \infty$

However, if we replace n^2 by n in the definition of f_n then we get

$$\int_0^1 f_n(x)dx = \int_0^1 n^2x(1-x^2)^n dx = \frac{n}{2n+2}$$

and

$$\lim_{n\to\infty}\int_0^1 f_n(x)dx = \frac{1}{2} \neq 0 = \int_0^1 \left[\lim_{n\to\infty} f_n(x)\right] dx$$

Uniform Convergence Definition

Definition (7.7)

We say that a sequence of functions $\{f_n\}, n = 1, 2, 3, \ldots$ converges *uniformly on E* to a function f if for every $\epsilon > 0$ there is an integer N such that $n \geq N$ implies

$$|f_n(x) - f(x)| \leq \epsilon$$

for all $x \in E$

It is clear that every uniformly convergent sequence converges pointwise.

Uniform Convergence Definition

Definition (7.7)

We say that a series $\sum f_n$ converges uniformly on E if the sequence of partial sums $\{s_n\}$ defined by

$$s_n(x) = \sum_{i=1}^{n} f_i(x)$$

converges uniformly on E.

Cauchy Condition

Theorem (7.8)

The sequence of functions $\{f_n\}$ defined on E, converges uniformly on E if and only if for every $\epsilon > 0$ there exists an integer N such that $m \geq N$, $n \geq N$, $x \in E$ implies

$$|f_n(x) - f_m(x)| \leq \epsilon$$

Cauchy Condition

Theorem (7.9)

Suppose

$$\lim_{n\to\infty} f_n(x) = f(x) \qquad (x \in E)$$

Put

$$M_n = sup_{x\in E}|f_n(x) - f(x)|$$

Then $f_n \to f$ *uniformly on* E *if and only if* $M_n \to 0$ *as* $n \to \infty$

Cauchy Condition For Series

Theorem (7.10)

Suppose $\{f_n\}$ *is a sequence of functions defined on* E *and suppose*

$$|f_n(x)| \leq M_n \qquad (x \in E, n = 1, 2, 3 \cdots)$$

Then $\sum f_n$ *converges uniformly on* E *if* $\sum M_n$ *converges*