Integral of Vector Valued Functions

Definition (6.23)

Let $f_1, \ldots, f_k$ be real functions on $[a, b]$ and set $\mathbf{f} = (f_1, \ldots, f_k)$ be the corresponding mapping of $[a, b]$ into R^k. If α increases monotonically on $[a, b]$, to say that $\mathbf{f} \in \mathcal{R}(\alpha)$ means that $f_j \in \mathcal{R}(\alpha)$ for $j = 1, \ldots, k$. If this is the case we define

$$\int_a^b \mathbf{f} d\alpha = \left(\int_a^b f_1 \ d\alpha, \ldots, \int_a^b f_k \ d\alpha \right)$$

Vector Valued Fundamental Theorem of Calculus

Theorem (6.24)

If $\mathbf{f}$ *and* $\mathbf{F}$ *map* $[a, b]$ *into* $\mathbb{R}^k$, $\mathbf{f} \in \mathcal{R}$ *on* $[a, b]$ *and if* $\mathbf{F}' = \mathbf{f}$ *then*

$$\int_a^b \mathbf{f}(t)\, dt = \mathbf{F}(b) - \mathbf{F}(a)$$

Vector Valued Integration and Absolute Value

Theorem (6.25)

If $\mathbf{f}$ *maps* $[a, b]$ *into* $\mathbb{R}^k$ *and if* $\mathbf{f} \in \mathcal{R}(\alpha)$ *for some monotonically increasing function* α *on* $[a, b]$ *then* $|\mathbf{f}| \in \mathcal{R}(\alpha)$ *and*

$$\left| \int_a^b \mathbf{f} \, d\alpha \right| \leq \int_a^b |\mathbf{f}| \, d\alpha$$

Definition of Rectiviable Curves

Definition (6.26)

A continuous mapping γ of an interval $[a, b]$ into $\mathbb{R}^k$ is called a *curve* in $\mathbb{R}^k$. To emphasize the parameter interval $[a, b[$ we may also say γ is a curve on $[a, b]$

- If γ is one-to-one, γ is called an *arc*
- If $\gamma(a) = \gamma(b)$, γ is said to be a *closed curve*.

Notice that a curve is the *mapping* not just the range of the curve.

Rectifiable Curves and Partitions

Definition (6.26)

Suppose γ is a curve in $\mathbb{R}^k$ on $[a, b]$ and and $P = \{x_0, \ldots, x_n\}$ is a partition of $[a, b]$. We then define

$$\Lambda(P, \gamma) = \sum_{i=1}^{m} |\gamma(x_i) = \gamma(x_{i-1})|$$

So $\Lambda(P, \gamma)$ is the length of the polygon whose vertices are the images of the points of the partition.

Rectifiable Curves and Partitions

Definition (6.26)

We therefore define the *length* of γ to be

$$\Lambda(\gamma) = \sup \Lambda(P, \gamma)$$

where the supremum is taken over all partitions of $[a, b]$.

If $\Lambda(\gamma) < \infty$ then we say γ is *rectifiable*.

Rectifiable and Continuous Curves

Theorem (6.27)

If γ' is continuous on $[a, b]$, then γ is rectifiable and

$$\Lambda(\gamma) = \int_a^b |\gamma'(t)| \, dt$$