Properties of Integrals

Theorem (6.12)

(a) *If* $f_1, f_2 \in \mathcal{R}(\alpha)$ *on* $[a, b]$ *then*

$$f_1 + f_2 \in \mathcal{R}(\alpha)$$

$cf \in \mathcal{R}(\alpha)$ *for every constant* c*, and*

$$\int_a^b (f_1 + f_2)\ d\alpha = \int_a^b f_1\ d\alpha + \int_a^b f_2\ d\alpha$$

and

$$\int_a^b cf\ d\alpha = c\int_a^b f\ d\alpha$$

Properties of Integrals

Theorem (6.12)

(b) *If* $f_1(x) \leq f_2(x)$ *on* $[a, b]$ *then*

$$\int_a^b f_1 d\alpha \leq \int_a^b f_2 \; d\alpha$$

(c) *If* $f \in \mathcal{R}(\alpha)$ *on* $[a, b]$ *and if* $a < c < b$ *then* $f \in \mathcal{R}(\alpha)$ *on* $[a, c]$ *and on* $[c, b]$ *and*

$$\int_a^c f \; d\alpha + \int_c^b f \; d\alpha = \int_a^b f \; d\alpha$$

Properties of Integrals

Theorem (6.12)

(d) *If* $f \in \mathcal{R}(\alpha)$ *on* $[a, b]$ *and if* $|f(x)| \leq M$ *on* $[a, b]$ *then*

$$\left| \int_a^b f \, d\alpha \right| \leq M[\alpha(b) - \alpha(a)]$$

(e) *If* $f \in \mathcal{R}(\alpha_1)$ *and* $f \in \mathcal{R}(\alpha_2)$ *then* $f \in \mathcal{R}(\alpha_1 + \alpha_2)$ *and*

$$\int_a^b f d(\alpha_1 + \alpha_2) = \int_a^b f d\alpha_1 + \int_a^b f d\alpha_2$$

if $f \in \mathcal{R}(\alpha)$ *and* c *is a positive constant, then* $f \in \mathcal{R}(c\alpha)$ *and*

$$\int_a^b f d(c\alpha) = c \int_a^b f d\alpha$$

Products/Absolute Values

Theorem (6.13)

Suppose $f, g \in \mathcal{R}(\alpha)$ *and* $[a, b]$ *then*

(a) $fg \in \mathcal{R}(\alpha)$

(b) $|f| \in \mathcal{R}(\alpha)$ *and* $\left|\int_a^b f \, d\alpha\right| \leq \int_a^b |f| \, d\alpha$

Step Function

Definition (6.14)

The *unit step function* I is defined by

$$I(x) = 0 \quad (x \leq 0)$$
$$1 \quad (x > 1)$$

Theorem (6.15)

If $a < s < b$ f *is bounded on* $[a, b]$, f *is continuous at* s *and* $\alpha(x) = I(x - s)$ *then*

$$\int_a^b f \; d\alpha = f(x)$$

Step Function and Sequences

Theorem (6.16)

Suppose $c_n \geq 0$ for $1, 2, 3, \ldots, \sigma c_n$ converges, $\{s_n\}$ is a sequence of distinct points in (a, b) and

$$\alpha(x) = \sum_{n=1}^{\infty} c_n I(x - s_n)$$

Let f be continuous on $[a, b]$. Then

$$\int_a^b f \; d\alpha = \sum_{n=1}^{\infty} c_n f(s_n)$$

Monotonically Increasing α

Theorem (6.17)

Assume α increases monotonically and $\alpha' \in \mathcal{R}$ on $[a, b]$. Let f be a bounded real function on $[a, b]$.

Then $f \in \mathcal{R}(\alpha)$ if and only if $f\alpha' \in \mathcal{R}$. In this case

$$\int_a^b f \, d\alpha = \int_a^b f(x)\alpha'(x)dx$$

Change of Variables

Theorem (6.19)

Suppose φ is a strictly increasing continuous function that maps an interval $[A, B]$ onto $[a, b]$. Suppose α is monotonically increasing on $[a, b]$ and $f \in \mathcal{R}(\alpha)$ on $[a, b]$. Define β and g on $[A, B]$ by

$$\beta(y) = \alpha(\varphi(y)) \qquad g(y) = f(\varphi(y))$$

Then $g \in \mathcal{R}(\beta)$ and

$$\int_A^B g \; d\beta = \int_a^b f \; d\alpha$$

Change of Variables

As a special case notice that if $\alpha(x) = x$, $\beta = \varphi$ and $\varphi' \in \mathcal{R}$ on $[A, B]$ then

$$\int_a^b f(x)dx = \int_A^B f(\varphi(y))\varphi'(y)dy$$