Definition of Riemann Integrals

Definition (6.1)

Let $[a, b]$ be a given interval. By a *partition* P of $[a, b]$ we mean a finite set of points $x_0, x_1, \ldots, x_n$ where

$$a = x_0 \leq x_1 \leq \cdots \leq x_{n-1} \leq x_n = b$$

We write $\Delta x_i = x_i - x_{i-1}$ for $i = 1, \ldots n$.

Definition of Riemann Integrals

Definition (6.1)

Now suppose f is a bounded real function defined on $[a, b]$. Corresponding to each partition P of $[a, b]$ we put

$$M_i = \sup f(x) \quad (x_{i-1} \le x \le x_i)$$

$$m_i = \inf f(x) \quad (x_{i-1} \le x \le x_i)$$

$$U(P, f) = \sum_{i=1}^{k} M_i \Delta x_i$$

$$L(P, f) = \sum_{i=1}^{k} m_i \Delta x_i$$

Definition of Riemann Integrals

Definition (6.1)

We put

$$\overline{\int_a^b} f dx = \inf U(P, f)$$

and we call this the *upper Riemann integral of f*.

We also put

$$\underline{\int_a^b} f dx = \sup L(P, f)$$

and we call this the *lower Riemann integral of f*.

Definition of Riemann Integrals

Definition (6.1)

If the upper and lower Riemann integrals are equal then we say f is Riemann-integrable on $[a, b]$ and we write $f \in \mathcal{R}$. We denote the common value, which we call the *Riemann integral of f on* $[a, b]$ as

$$\int_a^b f dx \text{ or } \int_a^b f(x) dx$$

Left and Right Riemann Integrals

If f is bounded then there exists two numbers m and M such that

$$m \leq f(x) \leq M \text{ if } (a \leq x \leq b)$$

Hence for every partition P we have

$$m(b-a) \leq L(P,f) \leq U(P,f) \leq M(b-a)$$

and so $L(P,f)$ and $U(P,f)$ both form bounded sets (as P ranges over partitions). Hence both $L(P,f)$ and $U(P,f)$ are defined for every bounded function f.

The question as to whether or not $L(P,f)$ equals $U(P,f)$ is harder.

Riemann-Stieltjes Integrals

Definition (6.2)

Let α be a monotonically increasing function on $[a, b]$ (since $\alpha(a)$ and $\alpha(b)$ are finite, it follows that α is bounded on $[a, b]$).

Corresponding to each partition P on $[a, b]$ we write

$$\Delta\alpha_i = \alpha(x_i) - \alpha(x_{i-1})$$

It is clear that $\Delta\alpha_i \geq 0$ for all i

Riemann-Stieltjes Integrals

Definition (6.2)

For any bounded real valued function we put

$$U(P, f, \alpha) = \sum_{i=1}^{k} M_i \Delta\alpha_i$$

$$L(P, f, \alpha) = \sum_{i=1}^{k} m_i \Delta\alpha_i$$

Left and Right Riemann-Stieltjes Integral

Definition (6.2)

We put

$$\overline{\int_a^b} f d\alpha = \inf U(P, f, \alpha)$$

and

$$\underline{\int_a^b} f d\alpha = \sup L(P, f)$$

where the inf and sup range over all partitions.

Left and Right Riemann-Stieltjes Integral

Definition (6.2)

If these are equal then we denote the common value by

$$\int_a^b f d\alpha \text{ or } \int_a^b f(x) d\alpha(x)$$

which we call the *Riemann-Stieltjes integral* (or just the *Stieltjes integral*) of f with respect to α on $[a, b]$ and we write $f \in \mathcal{R}(\alpha)$

Taking $\alpha(x) = x$ we see that the Riemann integral is a special case of the Riemann-Stieltjes integral.

However it is worth mentioning that in the general case α need not even be continuous.

Partitions

Definition (6.3)

We say that the partition P^* is a *refinement* of P if $P^* \supset P$ (that is, if every point of P is a point of P^*). Given two partitions P_1 and P_2 we say that P^* is their *common refinement* if $P^* = P_1 \cup P_2$.

Theorem (6.4)

If P^* is a refinement of P then

$$L(P, f, \alpha) \leq L(P^*, f, \alpha)$$

and

$$U(P^*, f, \alpha) \leq U(P, f, \alpha)$$

Integrability Condition

Theorem (6.5)

$$\underline{\int_a^b} f d\alpha \leq \overline{\int_a^b} f d\alpha$$

Theorem (6.6)

$f \in \mathcal{R}(\alpha)$ on $[a, b]$ if and only if for every $\epsilon > 0$ there exists a partition P such that

$$U(P, f, \alpha) - L(P, f, \alpha) < \epsilon$$

Properties of Integration

Theorem (6.7)

(a) *If* $U(P,f,\alpha) - L(P,f,\alpha) < \epsilon$ *then for every* P^* *a refinement of* P, $U(P^*,f,\alpha) - L(P^*,f,\alpha) < \epsilon$ *also holds.*

(b) *If* $U(P,f,\alpha) - L(P,f,\alpha) < \epsilon$ *and* $P = \{x_0,\ldots,x_n\}$ *and if* s_i, t_i *are arbitrary points in* $[x_{i-1},x_i]$ *then*

$$\sum_{i=1}^{n} |f(s_i) - f(t_i)|\Delta\alpha_i < \epsilon$$

(c) *If* $f \in \mathcal{R}(\alpha)$ *and the hypothesis of* (b) *holds then*

$$\left|\sum_{i=1}^{n} f(t_i)\Delta\alpha_i - \int_a^b f d\alpha\right| < \epsilon$$

Existence of Integration

Theorem (6.8)

If f is continuous on $[a, b]$ then $f \in \mathcal{R}(\alpha)$ on $[a, b]$.

Theorem (6.9)

If f is monotonic on $[a, b]$ and if α is continuous on $[a, b]$ then $f \in \mathcal{R}(\alpha)$ (we still assume of course that α is monotonic).

Existence of Integration

Theorem (6.10)

Suppose f is bounded on $[a, b]$, f has only finitely many points of discontinuity on $[a, b]$ and α is continuous at every point at which f is discontinuous. Then $f \in \mathcal{R}(\alpha)$.

Notice that if f and α have a common point of discontinuity then f need not be in $\mathcal{R}(\alpha)$.

Existence of Integration and Composition

Theorem (6.11)

Suppose $f \in \mathcal{R}(\alpha)$ *on* $[a, b]$, $m \leq f(x) \leq M$ *for* $x \in [a, b]$, φ *is continuous on* $[m, M]$ *and* $h(x) = \varphi(f(x))$ *on* $[a, b]$. *Then* $h \in \mathcal{R}(\alpha)$ *on* $[a, b]$.