Taylor's Theorem

Theorem (5.15)

Suppose f is a real function on $[a, b]$ n is a positive integer, $f^{(n-1)}$ is continuous on $[a, b]$, $f^{(n)}(t)$ exists for every $t \in (a, b)$. Let α, β be distinct points of $[a, b]$, and define

$$P(t) = \sum_{k=0}^{n-1} \frac{f^{(k)}(\alpha)}{k!}(t-\alpha)^k$$

Then there exists a point x between α and β such that

$$f(\beta) = P(\beta) + \frac{f^{(n)}(x)}{n!}(\beta-\alpha)^n$$

Taylor's Theorem

For $n = 1$ this is just the mean value theorem. In general the theorem shows that f can be approximated by a polynomial of degree $n - 1$ and allows us to estimate the error if we know bounds on $|f^{(n)}(x)|$.

Differentiation of Complex Functions

Definition (5.16)

Our definition of differentiation applies to complex valued functions defined on $[a, b]$ If

$$f(t) = f_1(t) + if_2(t)$$

where f_1 and f_2 are real valued then we have

$$f'(t) = f_1'(t) + if_2'(t)$$

Also f is differentiable at x if and only if both f_1 and f_2 are.

Differentiation of Vector Valued Functions

Definition (5.16)

Let $\mathbf{f}$ be defined on $[a, b]$ taking values in $\mathbb{R}^n$. For any $x \in [a, b]$ $\mathbf{f}'(x)$ is the point, if there is one, for which

$$\lim_{t \to x} \left| \frac{\mathbf{f}(t) - \mathbf{f}(x)}{t - x} - \mathbf{f}'(x) \right| = 0$$

If $\mathbf{f} = (f_1, \ldots, f_n)$ with each f_1 a real valued function then

$$\mathbf{f}'(x) = (f_1'(x), \ldots, f_n'(x))$$

and $\mathbf{f}$ is differentiable at x if and only if f_i is differentiable at x for all $1 \leq i \leq n$.

Differentiation and Operations

Theorem

Suppose $\mathbf{f}$ *and* $\mathbf{g}$ *are defined on* $[a, b]$*, are vector valued and differentiable at a point* $x \in [a, b]$*. Then* $\mathbf{f} + \mathbf{g}$*,* $\mathbf{f} \cdot \mathbf{g}$ *are differentiable at* x*, and*

(a) $(\mathbf{f} + \mathbf{g})'(x) = \mathbf{f}'(x) + \mathbf{g}'(x)$

(b) $(\mathbf{f} \cdot \mathbf{g})'(x) = \mathbf{f}'(x) \cdot \mathbf{g}(x) + \mathbf{f}(x) \cdot \mathbf{g}'(x)$

Example

For real x define

$$f(x) = e^{ix} = \cos(x) + i\sin(x)$$

(This can be taken as the definition of complex exponentiation).
Then $f'(x) = ie^{ix} = -\sin(x) + i\cos(x)$

Then we have $f(2\pi) - f(0) = 1 - 1 = 0$ but $|f'(x)| = 1$ for all real x. Hence the mean value theorem fails in this case.

Example

On the segment $(0, 1)$ define $f(x) = x$ and

$$g(x) = x + x^2 e^{i/x^2}$$

Since $|e^{it}| = 1$ for all real t we see that

$$\lim_{x \to 0} \frac{f(x)}{g(x)} = 1$$

Next we have

$$g'(x) = 1 + \left\{ 2x - \frac{2i}{x} \right\} e^{i/x^2} \qquad (0 < x < 1)$$

so that

$$|g'(x)| \geq \left| 2x - \frac{2i}{x} \right| - 1 \geq \frac{2}{x} - 1$$

Example

Hence

$$\left|\frac{f'(x)}{g'(x)}\right| = \frac{1}{|g'(x)|} \leq \frac{x}{2-x}$$

and so

$$\lim_{x\to 0} \frac{f'(x)}{g'(x)} = 0$$

Hence in particular L'Hospital's Rule fails in this case.

Mean Value Theorem for Vector Value Functions

Theorem (5.19)

Suppose $\mathbf{f}$ *is a continuous mapping of* $[a, b]$ *into* $\mathbb{R}^n$ *and* $\mathbf{f}$ *is differentiable in* (a, b) *such that*

$$|\mathbf{f}(b) - \mathbf{f}(a)| \leq (b - a)|\mathbf{f}'(x)|$$