

# Math 104-006

## Chapter 12.10: Taylor and Maclaurin Series

# Outline For Today

- Error in approximations

# Taylor Series

- Suppose  $f(x)$  can be represented by a power series if  $|x-a| < R$ . I.e.

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \dots$$

- Then  $c_n = \frac{f^{(n)}(a)}{n!} (x-a)^n$

# Taylor Series Continued

Then

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n = f(a) + \frac{f'(a)}{1!} (x-a) + \frac{f''(a)}{2!} (x-a)^2 + \dots$$

When the series converges.

This is the **Taylor series of the function  $f$  at  $a$**

# Maclaurin Series

If  $a = 0$  then the Taylor series becomes

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + \frac{f'(0)}{1!} x + \frac{f''(0)}{2!} x^2 + \dots$$

We give this series a special name. It is called the **Maclaurin series**.

# Maclaurin Series of Cos(x)

- Lets find the Maclaurin series for  $f(x)=\cos(x)$

- We know  $f(x) = \cos(x) = f^{(4n)}(x)$

- $f'(x) = -\sin(x) = f^{(4n+1)}(x)$

- $f''(x) = -\cos(x) = f^{(4n+2)}(x)$

- $f'''(x) = \sin(x) = f^{(4n+3)}(x)$

- So  $f^{(4n)}(0) = 1$   $f^{(4n+1)}(0) = 0$

- $f^{(4n+2)}(0) = -1$   $f^{(4n+3)}(0) = 0$

# Maclaurin of Cos(x) Continued

So the Maclaurin series is

$$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$$

Further

$$\lim_{n \rightarrow \infty} \frac{x^{2n+2} / (2n+2)!}{x^{2n} / (2n)!} = \lim_{n \rightarrow \infty} \frac{x^2}{(2n+2)(2n+1)} = 0$$

So the series converges for all x.

# Maclaurin series of Sin(x)

- Lets find the Maclaurin series for  $f(x)=\sin(x)$

- We know  $f(x) = \sin(x) = f^{(4n)}(x)$

- $f'(x) = \cos(x) = f^{(4n+1)}(x)$

- $f''(x) = -\sin(x) = f^{(4n+2)}(x)$

- $f'''(x) = -\cos(x) = f^{(4n+3)}(x)$

- So  $f^{(4n)}(0) = 0$   $f^{(4n+1)}(0) = 1$

- $f^{(4n+2)}(0) = 0$   $f^{(4n+3)}(0) = 1$

# Maclaurin of sin(x) Continued

So the Maclaurin series is

$$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$$

Further

$$\lim_{n \rightarrow \infty} \frac{x^{2n+3} / (2n+3)!}{x^{2n+1} / (2n+1)!} = \lim_{n \rightarrow \infty} \frac{x^2}{(2n+3)(2n+2)} = 0$$

So the series converges for all x.

# Try An Example

What is a Maclaurin series of  $e^x$ ?

A) 
$$\sum_{n=0}^{\infty} \frac{x^{2n} + 1}{(2n + 1)!}$$

D) 
$$\sum_{n=0}^{\infty} (-1)^n \frac{x^{3n}}{(3n)!}$$

B) 
$$\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$$

E) 
$$\sum_{n=0}^{\infty} \frac{x^n}{n!}$$

C) 
$$\sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n!}$$

F) None of the above

# Try An Example

What is a Maclaurin series of  $e^x$ ?

A)  $\sum_{n=0}^{\infty} \frac{x^{2n} + 1}{(2n + 1)!}$

D)  $\sum_{n=0}^{\infty} (-1)^n \frac{x^{3n}}{(3n)!}$

B)  $\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$

E)  $\sum_{n=0}^{\infty} \frac{x^n}{n!}$

C)  $\sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n!}$

F) None of the above

# Maclaurin of $(1+x)^k$

$$f(x) = (1+x)^k$$

$$f(0)=1$$

$$f'(x) = k(1+x)^{k-1}$$

$$f'(0)=k$$

$$f''(x) = k(k-1)(1+x)^{k-2}$$

$$f''(0)=k(k-1)$$

$$f'''(x) = k(k-1)(k-2)(1+x)^{k-3}$$

$$f'''(0) = k(k-1)(k-2)$$

...

$$f^{(n)}(x) = k(k-1)\dots(k-n+1)(1+x)^{k-n}$$

$$f^{(n)}(0) = k(k-1)\dots(k-n+1)$$

# Maclaurin of $(1+x)^k$ Continued

So the Maclaurin series is

$$\begin{aligned}(1+x)^k &= \sum_{n=0}^{\infty} \binom{k}{n} x^n \\ &= \sum_{n=0}^{\infty} \left( \frac{k(k-1)\dots(k-n+1)}{n!} \right) x^n \\ &= 1 + kx + \left( \frac{k(k-1)}{2!} \right) x^2 + \left( \frac{k(k-1)(k-2)}{3!} \right) x^3 + \dots\end{aligned}$$

# Table of Series

$\frac{1}{1-x}$	$= \sum_{n=0}^{\infty} x^n$	$= 1 + x + x^2 + x^3 \dots$	$R = 1$
$e^x$	$= \sum_{n=0}^{\infty} \frac{x^n}{n!}$	$= 1 + x + \frac{x^2}{2!} + \dots$	$R = \infty$
$(1+x)^k$	$= \sum_{n=0}^{\infty} \binom{k}{n} x^n$	$= 1 + kx + \left( \frac{k(k-1)}{2!} \right) x^2 + \dots$	$R = 1$
$\sin(x)$	$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$	$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$	$R = \infty$
$\cos(x)$	$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$	$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$	$R = \infty$
$\tan^{-1}(x)$	$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)}$	$= x - \frac{x^3}{3} + \frac{x^5}{5} - \dots$	$R = 1$
$-\ln  1-x $	$= \sum_{n=0}^{\infty} \frac{x^n}{n}$	$= 1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \dots$	$R = 1$

# Taylor Polynomials (Exceptions)

- We have shown that if a function is equal to a power series around  $x=a$  then the series must be the Taylor series about  $a$ .

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$$

- However there are some functions which are not equal to their Taylor series.

# nth degree Taylor Polynomial

- We can approximate a function by the partial sums of the Taylor series.
- Then **nth degree Taylor polynomial of f at a** is

$$T_n(x) = \sum_{m=0}^n \frac{f^{(m)}(a)}{m!} (x-a)^m = f(a) + \frac{f'(a)}{1!} (x-a) + \frac{f''(a)}{2!} (x-a)^2 + \dots$$

- We also define the **remainder** to be

$$R_n(x) = f(x) - T_n(x)$$

So  $f(x) = T_n(x) + R_n(x)$

# Remainder

- We know that

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n = \lim_{n \rightarrow \infty} T_n(x) = \lim_{n \rightarrow \infty} f(x) - R_n(x) = f(x) - \lim_{n \rightarrow \infty} R_n(x)$$

- So  $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$  if and only if  $\lim_{n \rightarrow \infty} R_n(x) = 0$

# Taylor's Inequality

- If  $|f^{(n+1)}(x)| \leq M$  for  $|x-a| \leq d$  then the remainder,  $R_n(x)$ , of the Taylor series satisfies the inequality.

- $|R_n(x)| \leq \frac{M}{(n+1)!} |x-a|^{n+1}$  for  $|x-a| \leq d$

# Taylor's Inequality For $\sin(x)$

- We know that if  $f(x) = \sin(x)$  then
- $|f^{(n)}(x)| \leq 1$  for all  $x$ . So
- $|R_n(x)| \leq \frac{1}{(n+1)!} |x|^{n+1}$  for all  $x$  and hence
- $\lim_{n \rightarrow \infty} |R_n(x)| = 0$  and hence

$$\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$$

# Taylor's Inequality For $\cos(x)$

- We know that if  $f(x) = \cos(x)$  then
- $|f^{(n)}(x)| \leq 1$  for all  $x$ . So
- $|R_n(x)| \leq \frac{|x|^{n+1}}{(n+1)!}$  for all  $x$  and hence
- $\lim_{n \rightarrow \infty} |R_n(x)| = 0$  and hence

$$\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$$

# Taylor's Inequality For $e^x$

- We know that if  $f(x) = e^x$  then
- $|f^{(n)}(x)| \leq e^d$  for all  $x \leq d$ . So
- $|R_n(x)| \leq \frac{e^d}{(n+1)!} |x|^{n+1}$  if  $|x| \leq d$  and hence
- $\lim_{n \rightarrow \infty} |R_n(x)| = 0$  for all  $x$  so for all  $x$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$