Questions From Math 170: Ideas in Mathematics (Spring 2007)

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1 Probability

- What is the probability that two people in a room of n will have the same birthday?
- Independent Outcomes
  - What does it mean for two outcomes to be independent?
  - What is the probability of two independent outcomes both happening (in terms of the probabilities of the individual outcomes).
  - What is the probability that neither of the independent outcomes will happen?
- Dependent Events
  - What does it mean for two outcomes to be dependent (i.e. not independent)?
  - Consider two possible outcomes, Outcome A and Outcome B. What is the probability of (Outcome A OR Outcome B) in term of the probabilities of Outcome A, Outcome B and (Outcome A AND Outcome B)?

  Repeated Events Suppose I have a single event (like an experiment or flip of a coin) and I have the probability of a given outcome doesn't depend on previous instances of the event.

  - Suppose I repeat this event n times and each time the probability of Outcome A is p. What is the probability of Outcome A never happening?

  - What is the probability of Outcome A happening at least once?

- How can probabilities be thought of in terms of a bag of marbles?

2 Chaos

- What is the (mathematical) definition of Chaos?
- What is a Discrete Dynamical System?
- How do you find the equilibrium of a Discrete Dynamical System?
- What does it mean for an equilibrium to be stable? Unstable? Unstable from the left (or negative) side? Unstable from the right (or positive side)?
- How do you determine if an equilibrium is stable or unstable?
- What are Spider Diagrams?
- Can you have an equilibrium which is unstable but where a small change from the equilibrium doesn't cause the system to go to infinity?

3 Complex Numbers

- What is a complex number?
- What is the norm of a complex number? How do you calculate it? Why is the norm always positive or 0?

- What is the complex conjugate of a complex number? What is the relationship between the complex conjugate of a number and it's norm?
- What is the fundamental theorem of algebra? Is there any relationship between complex conjugation and factoring polynomials?
- What is a Julia set?
- What is a Mandlebrot set?
- What is the relationship between Julia sets and Mandlebrot sets?

4 Fractals

- What is a fractal?
- How do you calculate the dimension of a shape?

5 Functions

- What is the definition of a function?
- What is the definition of the inverse of a function?
- What is a bijective function?
- What is a 1-to-1 function?
- What is a necessary and sufficient condition for a function to have an inverse?

6 Exponents/Logarithms

- What are the laws of exponents?
- What are the laws of logarithms?
- What is the connection between exponents and logarithms?
- What is the definit inverse of a function
- What is the number e? What is the natural logarithm?

7 Induction/Recursion

- What is a recursive definition?
- What is proof by induction?

8 Fibonacci Numbers and the Golden Ratio

- What is the definition of the Fibonacci numbers?
- What is the Golden Ratio (φ)?
- What is the connection between the Golden Ration and the Fibonacci Numbers?
- What is the connection between the Golden Ration and rectangles? And a sense of beauty?
- What is the continued fraction representation of the golden ratio?

- What quadratic polynomial does the golden ration satisfy?

9 Continued Fractions

- What is a continued fraction?
- What does $[a_0, a_1, \dots]$ represent?
- How can $[a_0, a_1, \dots]$ be considered as the limit of a sequence of rational numbers?
- Can every real number be represented as a continued fraction?

10 Cauchy Sequences and Real Numbers

- What is a Cauchy Sequence?
- When do two Cauchy Sequences equal each other?
- How do we add Cauchy Sequences?
- What is a Rational Number? What is an Irrational Number?
- What is the relationship between an infinite decimal expansion (or an infinite continued fraction) and a Cauchy Sequence?

11 Equivalence Classes

- What is an equivalence relation?

- What does it mean to "Mod Out" by an equivalence relation?
- If A is a set and $\sim$ is an equivalence relation, what conditions must be satisfied for an operation on $A/\ $ to make sense? (e.g. $+$ on Cauchy Sequences, or $\times$ in modular arithmetic)

12 Pascal's Triangle

- Pascal's Triangle
  - What is Pascal's Triangle?
  - What is the recursive definition?
  - How can you quickly write out the first n rows?
  - What is the sum of all the elements in the nth row?
  - What is the connection to the Serpinski triangle?
- n choose k
  - What is the relationship between Pascal's Triangle and $\binom{n}{k}$?
  - What does $\binom{n}{k}$ represent in the real world?
  - What is a representation of $\binom{n}{k}$? in term of factorials?
  - What is the relationship between $\binom{n}{k}$? and $\binom{n}{n-k}$?

  - If p is prime, why must p divide $\begin{pmatrix} p \\ k \end{pmatrix}$?

- Binomial Coefficients

  - What is the binomial theorem?
  - What is the connection between Pascal's Triangle and binomial coefficients?
  - What is the coefficient of $x^k y^{n-k}$ in $(x+y)^n$?

13 Prime Numbers

- Primes

  - What is a prime number? What is a composite number?
  - What is the fundamental theorem of arithmetic? Why does it fail for $\mathbb{Z}[\sqrt{-5}]$?
  - How do you prove $\sqrt{p}$ is irrational for all primes p?

- Relatively Prime

  - What is the greatest common divisor of two numbers (gcd)?
  - How do you find the gcd of two numbers (using Euclid's Algorithm)?
  - What does it mean for two numbers to be relatively prime?
  - What is $\phi(n)$?

  - How do you calculate the number of numbers less than n which are relatively prime to n?

14 Modular Arithmetic

- What is modular arithmetic?
- How do you define $+, \times, /, -$?
- What is a multiplicative inverse?
- When does an element of the integers mod n have a multiplicative inverse?
- What is Euclid's Algorithm? How does it help you find multiplicative inverses?
- What is Fermat's Little Theorem?
- What is Euler's Theorem?

15 Cryptography

- Substitution Codes
  - What is a substitution code?
  - What does one need to encode/decode a message using a substitution code?

  - How can a statistical analysis of large amount of coded data be used to break a substitution code?
  - What are the three most common letters?

- Permutation Code
  - What is a permutation code?
  - How do you encrypt/decrypt a message with such a code?
  - How can you use large amounts of coded data to break the code?

- Le Chiffre Indechiffrable (Vigenere Cypher)
  - What is the Vigenere Cypher?
  - How do you encrypt/decrypt a message using it?
  - How can you use large amounts of coded data to break the code?

- Public Key Cryptography
  - What is public key cryptography?
  - How do you encrypt/decrypt a message using it?
  - What are the advantages/disadvantages of it?
  - When can you find the root of a number mod n? How do you find such a root (when you can)?

16 Computable Sets

- What is an algorithm?

- What is the Church-Turing Thesis?
- What is a primitive recursive function?
- What is a computable function?
- What is a computably enumerable function?
- What is the difference between primitive recursive functions, computable functions and computably enumerable functions?
- What is the diagonal argument for primitive recursive functions?
- Why doesn't it work on computable functions? (i.e. to find an algorithm for a non-computable function)
- Why doesn't it work on computably enumerable functions? (i.e. for an algorithm for a non-computably enumerable function)
- What is a universal computable function?
- How can the computably enumerable functions be encoded by natural numbers? What is an example of such an encoding?
- What is an oracle (in reference to computable functions)?
- What is an example of an non-computably enumerable function?