Questions From Math 170: Ideas in Mathematics (Spring 2007)

by Nathanael Leedom Ackerman

March 18, 2007

1 Probability

(1) If there are 10 people in a room what is the probability that two of them share a birthday?

(2) If there are 10 people in a room what is the probability that two of them were born in the same month?

(3) How many people do we need in a room for there to be at least a 70% chance that one of them has March 4 as their birthday?

(4) If 100 people are in a room, what is the probability that no two share a birthday?

A jar contains 5 red marbles, 4 blue marbles and 3 green marbles. Three people, Ron, Bill, and Gene each choose a marble from the bag in succession (in that order).

(5) What is the probability that Ron choose a red marble, Bill chose a blue marble and Gene choose a green marble? Does the order matter?

(6) What is the probability that all three will choose red marbles?

(7) What is the probability that they will all chose different colors?

(8) How many times must I flip a fair coin for there to be a $< 1\%$ chance of all the flips coming out the same way?

A fair 6 sided die is rolled until a 6 appears.

(9) What is the probability it will take exactly 6 tries?

(10) What is the probability it will take at east 6 tries?

Two 6 sided dice are rolled.

(11) What is the probability that the two dice will have different values?

(12) What is the probability that the sum of the values of the dice is 2? 3? 4? 5? 6? 7? 8? 9? 10? 11? 12?

(13) There is a bucket that has 3 bags. One bag has 2 gold coins inside, another has 2 silver coins, and the third bag has one gold and one silver. Dr. Ackerman pulls out one of the bags, and then without looking, pulls one of the coins out of the bag to reveal it to be gold. So the bag he has in hand certainly is not the bag with 2 silver coins. It is either the bag with 2 gold coins or the bag with one gold and one silver. What is the probability that the bag he is holding is the bag with two gold coins in it?

2 Discrete Dynamical Systems and Chaos

Let $P_{n+1} = P_n^3$

(1) What are the equilibrium points?

(2) Are they stable or unstable from the right? From the left?

(3) If $P_0 = 1.1$ what is P_10 to 3 decimal places?

(4) If $P_n = f(n, P_0)$ what is f? (i.e. what is a function that describes P_n in terms of n and the initial value?)

Let $P_{n+1} = -P_n^2 + P_n - 4$

(5) What are the equilibrium points?

(6) Are they stable or unstable from the right? From the left?

(7) If $P_0 = 2.1$ what is P_4 to 3 decimal places?

Let $P_{n+1} = \sqrt{P_n}$, where $P_0 \geq 0$ and the square root is the positive square root.

(8) What are the equilibrium points?

(9) Are they stable or unstable from the right? From the left?

(10) If $P_0 = 0.9$ what is P_4 to 3 decimal places?

3 Complex Numbers

(1) Let $f(x) = x^4 + 5x^2 + 4$. Knowing that i and $2i$ are roots of $f(x)$, factor $f(x)$ into monomials (i.e. terms of degree 1)

(2) Let $f(x) = x^4 + 2x^3 + 10x^2 - 6x + 65$. Notice that $f(1 + 2i) = 0$ and $f(-2 - 3i) = 0$. Find all real numbers a, b such that $f(a + bi) = 0$.

(3) What is the norm of $3 + 4i$? $12 + 5i$? $1 + 1i$? $12 - 5i$?

(4) Let a, b be real numbers with the norm of $a + bi = x$, the norm of $-a + bi = y$ and the norm of $-a - bi = z$. List x, y, z in order (noting all equalities)

4 Fractals

THESE ARE COMING

5 Real Numbers

(1) Give an expression for $[2, 2, 2, 2, \dots]$ in terms of fractions and square roots.

(2) Give an expression for $[2, 3, 2, 3, \dots]$ in terms of fractions and square roots.

(3) Give an expression for $[2, 4, 2, 4, \dots]$ in terms of fractions and square roots.

(4) Let

- $a_n = 0.11 \dots 9$ where there are n 1's.
- $b_n = 0.11 \dots 211 \dots 1$ where there are two sequences of n 1's.
- $c_n = 0.11 \dots 1$ where there are n 1's.

Notice $\{a_n\}, \{b_n\}, \{c_n\}$ are all Cauchy Sequences. What is the order of these three sequences?

6 Equivalence Class

(4) Let $X = \{(n, m) : n \in \boldsymbol{N}, m \in \boldsymbol{N}, n \leq 5, m \leq 5\}$ and say $(a, b) \sim (c, d)$ if $a \times b = c \times d$. How many elements does $X/\sim$ have?

(5) Let $(\mathbb{Z}/3)[x] = \{\sum_{i=1}^{n} a_i x^i : a_i \in \mathbb{Z}/3\}$ be the collection of polynomials in x with coefficients in the $\mathbb{Z}/3$. (for example $x^3 + 2x \in (\mathbb{Z}/3)[x]$). Let $\sum_{i=1}^{n} a_i x^i \sim \sum_{i=1}^{n} b_i x^i$ if $\sum_{i=1}^{n}(a_i - b_i)x^i = f(x)x^3$ for some $f(x) \in (\mathbb{Z}/3)[x]$. How many elements does $(\mathbb{Z}/3)[x]/\sim$ have? (Note that any term with an exponent greater than or equal to 3 is ~ 0)

(6) Let $(\mathbb{Z}/4)[x] = \{\sum_{i=1}^{n} a_i x^i : a_i \in \mathbb{Z}/4\}$ be the collection of polynomials in x with coefficients in the $\mathbb{Z}/4$. (for example $2x^4 + 3x^2 \in (\mathbb{Z}/4)[x]$). Let $\sum_{i=1}^{n} a_i x^i \sim \sum_{i=1}^{n} b_i x^i$ if $\sum_{i=1}^{n}(a_i - b_i)x^i = f(x)x^4$ for some $f(x) \in (\mathbb{Z}/4)[x]$. How many elements does $(\mathbb{Z}/4)[x]/\sim$ have? (Note that any term with an exponent greater than or equal to 3 is ~ 0)

7 Pascal's Triangle, Binomial Formula

Recall that the first 3 rows of Pascal's triangle are

Row 0		1		
Row 1	1		1	
Row 2	1	2		1

With the first 1 being the 0th element.

(1) Find the 5th element of the 11th row of Pascal's Triangle.

(2) Find the 7th element of the 9th row of Pascal's Triangle.

(3) What is the coefficient of x^4 in $(1 + 2x)^6$? In $(2 + x)^6$

(4) What is the coefficient of x^6 in $(1 + 3x)^8$? In $(3 + x^2)^8$

(5) What is the $\begin{pmatrix} 10 \\ 7 \end{pmatrix}$? $\begin{pmatrix} 5 \\ 2 \end{pmatrix}$? $\begin{pmatrix} 7 \\ 3 \end{pmatrix}$?

(6) If there are 6 people in a group, John, Tim, Mike, Alice, Jen and Mary, how may ways are their to separate them into two groups each with 3 people in them?

8 Greatest Common Divisor

Let $x = 1925$, $y = 2520$

(1) What is the greatest common divisor of x and y ($gcd(x, y)$)?

(2) Find integers a, b such that $ax + by = gcd(x, y)$.

Let $x = 20449$, $y = 16170$

(3) What is the greatest common divisor of x and y ($gcd(x, y)$)?

(4) Find integers a, b such that $ax + by = gcd(x, y)$.

Let $x = 3125$, $y = 3080$

(5) What is the greatest common divisor of x and y ($gcd(x, y)$)?

(6) Find integers a, b such that $ax + by = gcd(x, y)$.

9 Relatively Prime

(1) How many numbers less than $2^4 \cdot 3^4 \cdot 5^4$ which are relatively prime to $2^4 \cdot 3^4 \cdot 5^4$?

(2) How many numbers less than $2^2 \cdot 3^2 \cdot 11$ share a factor with $2^2 \cdot 3^2 \cdot 11$?

(3) How many elements of $\mathbb{Z}/2^4 \cdot 3^3$ have inverses?

(4) What is $\phi(2^4)$? $\phi(3^4)$? $\phi(2^4 \cdot 3^4)$?

10 kth Roots

(1) Find values of a such that,

$$\begin{array}{lll} a^5 = 5 \bmod 12, & a^5 = 7 \bmod 12, & a^5 = 11 \bmod 12 \\ a^7 = 5 \bmod 12, & a^7 = 7 \bmod 12, & a^7 = 11 \bmod 12 \\ a^11 = 5 \bmod 12, & a^11 = 7 \bmod 12, & a^11 = 11 \bmod 12 \end{array}$$

11 Cryptography

A	B	C	D	E	F	G	H	I	J	K	L	M
1	2	3	4	5	6	7	8	9	10	11	12	13
N	O	P	Q	R	S	T	U	V	W	X	Y	Z
14	15	16	17	18	19	20	21	22	23	24	25	26

(1) Use a cyclic code shifted by 6 to encode?
MONDAY IS THE BEST DAY

(2) Use a cyclic code shifted by 6 to decode?
TU ZAKYJGE OY ZNK HKYZ JGE

(3) Use a cyclic code shifted by 25 to encode?
I LIKE WEDNESDAY

(4) Use a cyclic code shifted by 25 to decode?
SGTQRCZXR ZQD SGD VNQRS

(5) Use a permutation code in blocks of 9 and permuted by 3 to encode?
THANK GOD IT IS FRIDAY

(6) Use a permutation code in blocks of 9 and permuted by 3 to decode?
SUAARYTDS AGARRTEE!

(7) Use a Vigenere cipher with key of DAY to encode?
SUNDAY IS THE LAST DAY

(8) Use a Vigenere cipher with key of DAYS to decode?
D AYSD AYSDAYSD

12 Recursive Functions

Let $p(x, y) = x + y$.

(1) Let $t(n) = Succ(p(n, n))$. What is $t(1)$? $t(2)$? $t(3)$? What is a general form for $t(n)$?

(2) Let $h(n)$ be defined by primitive recursion with
Base Function: $h(0) = f = 0$
Inductive Function: $h(n{+}1) = g(h(n), n) = p(\pi_1(h(n), n),\ t(\pi_2(h(n), n)))$
What is $h(1)$? $h(2)$? $h(3)$? $h(4)$? What is a general form for $h(n)$?