Lecture Notes Math 104: Calculus I (Spring 2007)

by Nathanael Leedom Ackerman

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1 TALK SLOWLY AND WRITE NEATLY AND BIG!!

2 Volume

First off if anyone did not get an e-mail from me it means that your e-mail account is having problems with the blackboard system. I am currently trying to do what I can on my end to fix this but you should also tell whoever runs your e-mail that this is a problem.

For those of you who didn't get the e-mail we are in A4 on Wednesdays. This is the only day we are in A4, the other days we are in A8. In this class we are going to learn how to calculate volumes. Because of that I will have to draw some 3-D shapes. So I apologize in advance for my drawing capabilities.

Go through the volume of a cylinder.

$$\text{Volume} = \text{Base Area} \times \text{Height}$$

Give examples of circular cylinder ($V = \pi r^2 h$) and rectangular box ($V = lwh$).

Say that just as we used rectangles to approximate the area under a curve, we will use cylinders to approximate the volume of a curve.

Draw a 3-D shape (looking like half a sphere)

If we let V be the volume of the shape, and we let $A(x^*)$ be the area of the cross section of the shape perpendicular to the x-axis at point x^*, then we see

$$V = \lim_{n\to\infty} \sum_{i=0}^{n} A(x_i)\Delta x$$

where Δx is the width of the interval we are using to approximate the volume by cylinders. So we have

Theorem 2.0.1. *The volume of a shape is*

$$V = \int_a^b A(x)dx$$

where $A(x^*)$ *is the area of the cross section of the shape perpendicular to the* x*-axis at* x^*

3 Rotating Areas

Lets look at an example

Example

Lets calculate the volume of the shape obtained by rotating the area under $f(x) = x^2$ from -1 to 1 about the x axis.

It isn't hard to see that the cross sectional area at x of this shape is just $\pi(x^2)^2$ or πx^4.

So the volume is

$$\int_{-1}^{1} \pi x^4 = \pi/5x^5\big]_{-1}^{1} = 2\pi/5$$

now it isn't hard to see that this procedure works if we rotate the area under any curve around the x-axis.

Theorem 3.0.2. *Let $f(x)$ be a non-negative function. Then the volume obtained by rotating the area under $f(x)$ on the interval $[a, b]$ around the x-axis is*

$$\int_{a}^{b} \pi f(x)^2 dx$$

Examples of circles with wholes cut out

Now that we know how to calculate volumes of areas rotated around the x-axis, we can combine that with tools from last lecture.

Example Suppose we want to rotate the area between $f(x) = x^3$ and $g(x) = x^2$ around the x-axis.

The fist thing we need to do is to draw a picture. And to do that we need to find out where the graphs meet and which is greater than which on each interval. It isn't hard to see that the two points of intersection are $x = 1, x = 0$.

Draw picture The first thing we need to do is to figure out what a cross section looks like.

Draw a cross section

It isn't hard to see that a cross section has area $\pi(g(x)^2 - f(x)^2)$ (because we have $f(x) \leq g(x)$ for all $x \in [0, 1]$). So we find that

$$V = \int_0^1 \pi(x^4 - x^6) = \pi/5x^5 - \pi/7x^7\Big]_0^1 = \pi/5 - \pi/7 = 2\pi/35$$

Just as in the previous case this technique generalizes to

Theorem 3.0.3. *Let $f(x), g(x)$ be functions such that $f(x) \geq g(x) \geq 0$. Then the volume obtained by rotating the area between $f(x)$ and $g(x)$ in the interval $[a, b]$*

around the x-axis is

$$\int_a^b \pi(f(x)^2 - g(x)^2)dx$$

4 Examples

Lets consider a shape where the cross section $\perp$ to the x-axis at point a is the area between the parabola's $x^2 - a^2$ and $a^2 - x^2$ and we are looking at it from 0 to b

Draw this (or at least try)

Now that we have draw this we need to work out what the area is at point a. So, we need to find the points of intersection of these two curves.

This is when $x^2 - a^2 = a^2 - x^2 or \pm a$. So the are of the cross section at point a is

$$\int_{-a}^{a} (a^2-x^2)-(x^2-a^2)dx = 4\int_0^a (a^2-x^2)dx = 4\left[a^2x - x^3/3\right]_0^a = 8/3a$$

Now that we know the area of the cross section at point a we need to find the volume of the curve from 0 to b. So the volume is

$$\int_0^b (8/3a^3)da = \left[2a^4/3\right]_0^b = 2b^4/3$$

Example is time If there is still time rotate the area between $y^4 = x$ and $y^2 = x$ about the y-axis.