Lecture Notes Math 371: Algebra (Fall 2006)

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November 16, 2006

TALK SLOWLY AND WRITE NEATLY!!

0.1 Cubic Equations

0.1.1 Story

Now we are going to consider the case of the cubic. However before we do it is interesting to consider a little bit of the history behind the solution.

Story

- In the 1500's one way mathematicians proved their skills was by challenging each other to solve problems.
- The first person known to have solved cubic equations algebraically was del Ferro but he told nobody.
- On his death bead del Ferro told his student Fior (who wasn't as good a mathematician).
- At the time there were several types of cubics because

negative numbers weren't used.

- Fior had only been shown how to solve $x^3 + ax = b$.
- Fior began to boast that he was able to solve cubics and a challenge between him and Tartaglia was arranged in 1535.
- At the time Tartaglia had been figured out how to solve a different type of cubic, those of the form $x^3 + ax^2 = b$.
- For the contest between Tartaglia and Fior, each man was to submit thirty questions for the other to solve.
- Fior submitted 30 of the same type. Which lead Tartaglia to figure out a solution. Tartaglia offered several different types which Fior.
- At the beginning Tartaglia couldn't solve Fiors problems, but then in the early hours of 13 February 1535

he discovered the general method to solve $x^3 + ax = b$. Tartaglia was then able to solve all thirty of Fior's problems in less than two hours.

- Fior had made little headway with Tartaglia's questions so Tartaglia was declared the winner but Tartaglia didn't take the prize as the honor of winning was enough.

- At this point Cardan enters the story. He was a public lecturer of mathematics at the Piatti Foundation in Milan.

- He heard about the contest and when it was clear that Tartaglia had discovered a general method he set out to try and find it for himself but was unable.

- A few years later, in 1539, Cardan contacted Tartaglia, requesting that the method could be included in a book he was publishing that year.

- Tartaglia declined stating his intention to publish his formula in a book of his own that he was going to write at a later date.
- Cardan accepted this and then asked to be shown the method, promising to keep it secret. Tartaglia, however, refused.
- An upset Cardan then wrote to Tartaglia challenging him to a debate but, at the same time, hinting that he had been discussing Tartaglia's brilliance with the governor of Milan who was one of Cardan's powerful patrons.
- When Tartaglia got this letter he radically changed his mind, realizing that association with the influential Milanese governor could be very rewarding and could maybe get him out of the modest teacher's job he then held.

- Tartaglia wrote back trying to get an invitation to meet the Milanese governor and Cardan invited him to come to the court.

- When he came to Milan, after much persuasion he agreed to tell Cardan his method if Cardan swore never to reveal it and to only ever write it down in code so that on his death bed no one could discover it from his papers.

- Cardan agreed and Tartaglia divulged his formula in the form of a poem, to help protect the secret, should the paper fall into the wrong hands.

- Tartaglia then left for home, but by the time he got back he felt he had made a mistake revealing his secret when Cardan wrote to him to try and continue their friendship Tartaglia rebuffed him and ridiculed the books Cardan had published since Tartaglia had left Milan.

- Based on Tartaglia's formula, Cardan and Ferrari, his assistant, made remarkable progress finding proofs of all cases of the cubic in solving the quartic equation.

- Tartaglia though made no move to publish his formula despite the fact that, it had become well known that such a method existed. Tartaglia probably wished to keep his formula in reserve for any upcoming debates.

- Cardan and Ferrari then travelled to Bologna in 1543 and learnt from della Nave that it had been del Ferro, not Tartaglia, who had been the first to solve the cubic equation.

- When he learned this Cardan felt that although he had sworn not to reveal Tartaglia's method he felt that nothing prevented him from publishing del Ferro's formula.

- In 1545 Cardan published a book which contained solutions to both the cubic and quartic equations and all of the additional work he had completed on Tartaglia's formula.

- In the book Del Ferro and Tartaglia are credited with their discoveries, as is Ferrari, and the story written down in the text.

- Tartaglia however was furious when he discovered that Cardan had disregarded his oath and his intense dislike of Cardan turned into a pathological hatred.

0.1.2 Solution

We will being by showing the ad hoc construction of Tartaglia. First lets start with a cubic

$$f(x) = x^3 + a_2x^2 + a_1x + a_0$$

and lets set $f(x) = 0$.

Computation is simpler if $a^2 = 0$ so lets substitute $x = x_1 - a_2/3$. We then see that it suffices to consider

$$f(x) = x^3 + px + q$$

where p, q are in our field F.

Next we want to make the substitution $x = u - v$ and we see

$$f(u - v) = (u^3 - v^3) - (3uv - p)(u - v) + q$$

Now $f(u - v) = 0$ if we have

$$(3uv - p) = 0$$

and

$$u^3 - v^3 + q = 0$$

And, as we have two variables we might hope to find solutions to these two equations.

We solve the first equation to get $v = p/3u$ and substituting into the second we get

$$3^3u^6 - p^3 + 3^3u^3q = 0$$

By some miracle this is a quadratic equation in u^3. Setting $y = u^3$ we see

$$3^3y^2 + 3^3qy - p^3 = 0$$

So

$$y = -\frac{q}{2} + \left(\sqrt{\frac{q}{2}}\right)^2 + \left(\sqrt{\frac{p}{3}}\right)^3$$

We then have

$$u = \sqrt[3]{-\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}}$$

and

$$v = \sqrt[3]{u^3 + q} = \sqrt[3]{\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}}$$

and $x = u - v$

0.1.3 Galois Theory

Now lets consider the Galois theory of an irreducible cubic polynomial. We can assume that

$$f(x) = x^3 + px + q$$

and let K be the splitting field of f over F and let $\alpha_1, \alpha_2, \alpha_3$ be the roots. We then know that

$$f(x) = x^3 + px + q = (x - \alpha_1)(x - \alpha_2)(x - \alpha_3)$$

expanding this equation we see that

$$\alpha_1 + \alpha_2 + \alpha_3 = 0$$

$$\alpha_1\alpha_2 + \alpha_2\alpha_3 + \alpha_1\alpha_3 = p$$

$$\alpha_1\alpha_2\alpha_3 = -q$$

The first of these show that α_3 is in the field $F(\alpha_1, \alpha_2)$. So we have a chain

$$F \subset F(\alpha_1) \subset F(\alpha_1, \alpha_2) = K$$

Lets call $L = F(\alpha_1)$. We then see that we break into two cases. Either $L = K$ or $L < K$.

Example Lets consider $f(x) = x^3 - 2$. The three roots of this polynomial are $\alpha_1 = \sqrt{2}$, $\alpha_2 = \zeta\sqrt{2}$, $\alpha_3 = \zeta^2\sqrt{2}$ where $\sqrt{2}$ is the real square root and $\zeta = e^{2\pi i/3}$

As $\sqrt{2} \in \mathbb{R}$ we see that if we let $F = \mathbb{Q}$ we are in the second case. That $\alpha_2 \notin L = F(\alpha_1)$.

However, if we let $F = \mathbb{Q}(\zeta)$ then we see that $\alpha_2 \in F(\alpha_1)$ and we are in the first case.

Factorization Now lets try and factor $f(x)$ in L. We know that

$$f(x) = (x - \alpha_1)h(x) = (x - \alpha_1)(x - \alpha_2)(x - \alpha_3)$$

where $h(x) = f(x)/(x - \alpha_1)$ in L (we know $\alpha_1 \in L$ and as it is a root, f factors in L).

Now if $h(x)$ factors in L then we see that $\alpha_2, \alpha_3 \in L$ and so $L = K$. So $L < K$ if and only if $h(x)$ is irreducible over L. And in this case the degree $[L(\alpha) : L] = 2$. And similarly, because $f(x)$ was assumed to be irreducible we have $[L : F] = 3$.

Hence we have $[K : F] = 3$ if $L = K$ and $[K : F] = 6$ if $L < K$.

Now as $G(K/F)$ must permute the roots $\{\alpha_1, \alpha_2, \alpha_3\}$ we find that $G(K/F) \subseteq S_3$. And what is more $G(K/F) = [K : F]$ by main theorem of Galois theory (because K is a splitting field).

So if $L < K$ we have $[K : F] = |G(K/F)| = 6 = |S_3|$ and so $G(K/F) = 6$.

If $L = K$ then $[K : F] = 3 = |G(K/F)| = 3$ so $G(K/F) = Z/(3)$.

Intermediary Fields Now lets determine the intermediate fields in the case $[K : F] = 6$ (in the case $[K : F] = 3$ there are none).

S_3 has 3 conjugate subgroups of order 2. And these correspond to the natural intermediary fields $F(\alpha_1), F(\alpha_2), F(\alpha_3)$. They are isomorphic but not equal.

However, we still have one more subgroup of S_3, and that is $\mathbb{Z}/(3)$. This subgroup must correspond to a field $F \subset L \subset K$. Further, as $G(K/L) = \mathbb{Z}/(3)$ we have

$[K : L] = 3$ and therefore $[L : F] = 2$.

So L is a quadratic extension of F and hence is generated by a square root. Further as there is only one subgroup of S_3 of order 3 we see that this is essentially the only element of F with a square root in K. Lets call this square root δ

Further we know that L is the fixed field of $\mathbb{Z}/(3)$. So an even permutation of the roots fixes δ while an odd permutation changes its sign. We therefore see that

$$\delta = (\alpha_1 - \alpha_2)(\alpha_2 - \alpha_3)(\alpha_3 - \alpha_1)$$

Further δ is not fixed by every element of $G(K/F)$ and so $\delta \notin F$. But δ^2 is fixed by every element of $G(K/F)$ and so $\delta^2 \in F$.

Discriminant

Definition 0.1.3.1. For any cubic polynomial $f(x) = (x - \alpha_1)(x - \alpha_2)(x - \alpha_3)$

$$D = (\alpha_1 - \alpha_2)^2(\alpha_2 - \alpha_3)^2(\alpha_3 - \alpha_1)^2$$

is called the Discriminant. It is the number which is zero if and only if f has a multiple root. In this way it is analogous to the discriminant for a quadratic polynomial.

It can be shown that the discriminant of $f(x) = x^3 + px + q$ is

$$D = -4p^3 - 27q^2$$

Theorem 0.1.3.2. *The discriminant of an irreducible cubic polynomial $f(x) \in F[x]$ is a square in F if and only if the degree of the splitting field is 3.*

Proof. If D is not a square in F then $\delta \notin F$ and hence $[F(\delta) : F] = 2$. Since $\delta \in K$, $[K : F]$ is divisible by 2 and hence is 6.

On the other hand if $\delta \in F$ then every element of the Galois group fixes δ. And, since odd permutations of the roots change the sign of δ they are not in G and so $G \neq S_3$. Hence $[K : F] = 3$ □

0.2 TODO

- Go through Lang's book on the same topics.