Lecture Notes Math 371: Algebra (Fall 2006)

by Nathanael Leedom Ackerman

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$$\boxed{\textbf{TALK SLOWLY AND WRITE NEATLY!!}}$$

0.1   Spectral Theorem on Hermitian Forms

Today we will prove the spectral theorem. Recall from last time $\boxed{\textbf{Spectral Theorem}}$ Let $\langle,\rangle$ be a bilinear form and let M be the matrix associated to $\langle,\rangle$ relative to an orthonormal basis B. Then if we change basis from B to another orthonormal basis B' then we see that the corresponding new matrix is

$$UMU^* = M'$$

where U is the change of basis. But as both M and M' are Id we have that U is unitary. So a change of basis orthonormal basis is done via unitary matrixes.

Theorem 0.1.0.1 (Spectral Theorem). *(a) Let T be a hermitian operator on a hermitian space V, $\langle,\rangle$.*

Then there is an orthonormal basis for V consisting of eigenvectors of T.

(b) Matrix form Let M be a hermitian matrix with respect to an orthonormal basis. There is a unitary matrix P such that PMP^* is a real diagonal matrix.

Proof. First, if $\dim(V) = 1$ then choose a vector $v \in V$ and normalize it so that $\langle v_1, v_1 \rangle = 1$. Then v_1 is obviously an eigenvector of T and so we are done.

Now assume the theorem is true for an hermitian space of dimension $n - 1$ and let V be a hermitian space of dimension n.

Now find an eigenvector v of T and normalize it so that $\langle v_1, v_1 \rangle = 1$ and v_1 is an eigenvector of T. Then find an

orthonormal basis for V containing v_1.

We then have that the matrix associated to T is

$$M = \begin{bmatrix} x & a_1 & \cdots & a_{n-1} \\ b_1 & & & \\ \vdots & & N & \\ b_{n-1} & & & \end{bmatrix}$$

But we know that T is hermitian and N is a hermitian operator on $Span(v_1)^{\perp}$ (Recall that $V = Span(v_1) \oplus Span(v_1)^{\perp}$). But we also know that because M is hermitian that $\overline{b_i} = a_i$ for all i.

However, v_1 has representation under the orthonormal

basis of

$$\begin{bmatrix} 1 \\ 0 \\ \ldots \\ 0 \end{bmatrix}$$

And so x is the eigenvalue of v_1 and $a_i = \overline{b_i} = 0$ for all i. In particular we have T takes $Span(v_1)^\perp$ onto itself.

But $Span(v)^\perp$ has dimension $n - 1$ and so there is an orthonormal basis $\{v_2, \ldots, v_n\}$ of $Span(v_1)^\perp$ consisting of eigenvectors for T. There fore $\{v_1, \ldots, v_n\}$ is an orthonormal basis for V consisting of eigenvectors of T.

$\square$

Theorem 0.1.0.2. *The eigenvalues of a hermitian operator are real numbers.*

Proof. Let a be an eigenvalue and let v be an eigenvector

for T such that $T(v) = av$. Then we have $\langle Tv, v \rangle = \langle v, Tv \rangle$. Hence

$$\langle av, v \rangle = \langle v, av \rangle$$

and, as $\langle v, v \rangle \neq =$ we therefore have $a = \bar{a}$. $\qquad\square$

0.2   Spectral Theorem on Bilinear Forms

The results for Hermitian Operators carry over to the case of real vector spaces.

> ### Results in the case of Bilinear Forms

The results we have proved have analogs for real symmetric matrixes.

Theorem 0.2.0.3. *Let $V, \langle , \rangle$ be a Euclidean space, let T be a linear operator and let M be the matrix of T with respect to an orthonormal basis for $V, \langle , \rangle$.*

(a) The matrix M is symmetric if and only if $\langle v, Tw \rangle = \langle Tv, w \rangle$ for all $v, w \in V$. If so T is called a

Symmetric Operator

(b) The matrix M is orthogonal if and only if $\langle v, w \rangle = \langle Tv, Tw \rangle$ for all $v, w \in V$. If so T is called an Orthogonal Operator

Proof. Identical to the Hermitian Space case. $\square$

Theorem 0.2.0.4 (Spectral Theorem (real case))**.**

(a) Let T be a symmetric operator on a real vector space V with a positive definite bilinear form. Then there is is an orthonormal basis of eigenvectors of T.

(b) Matrix form: Let M be a real symmetric $n \times n$ matrix relative to an orthonormal basis. There is an orthogonal matrix $P \in O_n(\mathbb{R})$ such that PMP^t is diagonal.

Proof. Because all real symmetric matrixes are hermitian matrixes we know that the eigenvalues of all real hermitian matrixes are real.

And, we therefore see that the proof of this is identical to the hermitian case. □

0.3   Spectral Theorem for Normal Operators

Previously we showed that any Hermitian matrix can be transformed into a real diagonal matrix D by a unitary matrix P such that

$$D = PMP^*$$

Now we ask the question, "What matrices can be transformed in this way into a diagonal matrix if we no longer require the diagonal to be real?" This turns out to have a nice answer.

$$\boxed{\textbf{Normal Operator}}$$

Definition 0.3.0.5. A matrix M is called <u>Normal</u> if it commutes with its adjoint. I.e.

$$MM^* = M^*M$$

or equivalently

$$(MM^*)^* = MM^*$$

(i.e. MM^* is hermitian.)

Lemma 0.3.0.6. *If M is normal and P is unitary then*

$$M' = PMP^*$$

is also normal

Proof. Assume M is normal. Then

$$M'M'^* = PMP^*(PMP^*)^* = PMP^*P^{**}M^*P^* = PMM^*P^* = PM$$

So PMP^* is normal. And the converse is by replacing P by P^*. $\qquad\square$

Definition 0.3.0.7. A linear operator $T : V \to V$ on a hermitian space V is a <u>Normal Operator</u> if the matrix associated to T relative to an orthonomral basis is normal.

Spectral Theorem for Normal Operators

Theorem 0.3.0.8. *A complex matrix M is normal if and only if there is a unitary matrix P such that PMP^* is diagonal.*

Proof. First note that diagonal matrixes commute and so diagonal matrixes are normal (because if D is diagonal then so is D^*).

By the previous lemma we see that if $PMP^* = D$ where D is diagonal then M is normal.

Now lets assume M is normal.

Notice that if M is a 1×1 matrix then M is diagonal and we are done.

Lets assume that the theorem is true for all $n - 1 \times n - 1$

matrixes and let M be an $n \times n$ matrix.

Let v_1 be an eigenvector of T which is normalized so $\langle v, v \rangle = 1$ just as in the case of the spectral theorem for hermitian operators.

Now extend v_1 to an orthonormal basis just as in the hermitian spectral theorem. We then see that M is changed to a matrix

$$M_1 = PMP^* = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & & & \\ \vdots & & N & \\ 0 & & & \end{bmatrix}$$

and we also have

$$M_1^* = PM^*P^* \begin{bmatrix} \overline{a_{11}} & 0 & \cdots & 0 \\ \overline{a_{12}} & & & \\ \vdots & & N^* & \\ \overline{a_{1n}} & & & \end{bmatrix}$$

Now as M is normal, so is M_1 and so $M_1 M_1^* = M_1^* M_1$.

Now the upper left element of $M_1 M_1^* = a_{11}\overline{a_11} + \cdots + a_{1n}\overline{a_{1n}}$, but the upper left element of $M_1^* M_1 = a_{11}\overline{a_11}$ and so we must have

$$a_{11}\overline{a_11} = a_{11}\overline{a_11} + \cdots + a_{1n}\overline{a_{1n}}$$

and so

$$0 = a_{12}\overline{a_12} + \cdots + a_{1n}\overline{a_{1n}}$$

But $x\overline{x} \geq 0$ for all $x \in \mathbb{C}$ and $x\overline{x} = 0$ if and only if

$x = 0$. So in particularly we have $a_{1i} = 0$ if $i > 1$. Hence

$$M_1 = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & N & \\ 0 & & & \end{bmatrix}$$

But we know that

$$M_1 M_1^* = \begin{bmatrix} a_{11}\overline{a_{11}} & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & NN^* & \\ 0 & & & \end{bmatrix}$$

and

$$M_1^* M_1 = \begin{bmatrix} a_{11}\overline{a_{11}} & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & N^*N & \\ 0 & & & \end{bmatrix}$$

and hence N is normal as well.

But then by induction there is a basis $\{v_2, \ldots, v_n\}$ of $Span(v_1)^\perp$ under which N is diagonal. Hence under the basis $\{v_1, \ldots, v_n\}$ of V we see that the matrix M associated to a linear transformation is diagonal.

But we also have that the change of basis is from one orthonormal basis to another and so the change of basis matrix must in fact be unitary. $\quad\square$

Corollary 0.3.0.9. *Every conjugacy class in the unitary group contains a diagonal matrix.*

Proof. This is because every unitary matrix U satisfies

$$UU^* = I = U^*U$$

and because the identity is hermitian. $\quad\square$

0.4 TODO

- Flush out the outline of math.

- Specifically figure out what to show other than the structure theorem.

- Come up with A BUNCH of examples (more than I can use) so that I don't run out of time.

- Go through Lang's book on the same topics.