Lecture Notes Math 371: Algebra (Fall 2006)

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TALK SLOWLY AND WRITE NEATLY!!

0.1 Hermitian Forms

Now that we have defined what a Euclidean space is we want to consider the analog for complex vector spaces. Specifically we want to look at what the dot product should be if our space is a complex vector space instead of a real one.

Well if we have $\langle x_1, \ldots, x_n \rangle \in \mathbb{C}^n$ then its length as an element of $\mathbb{R}^{2n}$ is

$$\sqrt{(a_1^2 + b_1^2) + \cdots + (a_n^2 + b_n^2)} = \sqrt{x_1\overline{x_1} + \cdots + x_n\overline{x_n}}$$

Where $\overline{x}$ is the complex conjugation of x.

Standard Hermitian Product

Definition 0.1.0.1. This suggests that the right generalization for dot product on complex vector spaces is the Standard Hermitian product

$$\langle X, Y\rangle = \overline{X}^t Y = \overline{x_1}y_1 + \cdots + \overline{x_n}y_n$$

Lemma 0.1.0.2. *Now this product has the following nice properties*

- $\langle X, Y\rangle$ *agrees with the dot product on the reals*
- $(\forall X \neq 0)\langle X, X\rangle$ *is a positive real.*

ne second variable) $\langle X, cY\rangle = c\langle X, Y\rangle$ *and* $\langle X, Y_1 + Y_2\rangle = \langle X, Y_1\rangle + \langle X, Y_2\rangle$

the first variable) $\langle cX, Y\rangle = \overline{c}\langle X, Y\rangle$ *and* $\langle X_1 + X_2, Y\rangle = \langle X_1, Y\rangle + \langle X_2, Y\rangle$

rmitian Symmetry) $\langle X, Y\rangle = \overline{\langle Y, X\rangle}$

Proof. Immediate. □

Now we can define the generalization of a bilinear form to complex vector spaces. **Hermitian Form**

Definition 0.1.0.3. We define a Hermitian Form on a complex vector space V to be an function

$$\langle , \rangle : V \times V \to V$$

such that $\langle , \rangle$ satisfies

- Linearity in the second variable
- Conjugate linearity in the first variable
- Hermitian symmetry

Just as in the real case we can define **Matrix Associated to**

Definition 0.1.0.4. Let $\langle , \rangle$ be a Hermitian form on a complex finite dimensional vector space V. Further let $B = \{v_1, \cdots , v_n\}$ be a basis for V. We then define the matrix A associated to the form re as

$$A = (a_{ij}) \text{ where } aij = \langle v_i, v_j \rangle$$

Just as in the case of bilinear forms we have

Lemma 0.1.0.5. *Let $\langle , \rangle$ be a Hermitian form on a complex finite dimensional vector space V. Further let $B = \{v_1, \cdots, v_n\}$ be a basis for V and let $v, w \in V$ be such that $v = BX$ and $w = BY$ then*

$$\langle v, w \rangle = \overline{X}^t AY$$

Proof. This is exactly the same as in the case of bilinear forms. □

There is one difference though between the Hermitian form and the bilinear form case. While in a bilinear form any matrix gave rise to a bilinear form, in the case of a Hermitian form we know that

$$a_{ij} = \langle v_i, v_j \rangle = \overline{\langle v_j, v_i \rangle} = \overline{a_{ji}}$$

which leads us to the following definition **Adjoint of a matrix**

Definition 0.1.0.6. Let A be a complex matrix then we define the adjoint of A to be

$$A^* = \overline{A}^t$$

Lemma 0.1.0.7. *We then have the following properties of adjoints*

$$(A + B)^* = A^* + B^*$$

$$(AB)^* = B^* A^*$$

$$(A^*)^{-1} = (A^{-1})^*$$

$$A^{**} = A$$

Proof. Immediate

Ask the class if they want to see any of these

□

Definition 0.1.0.8. A matrix is self-adjoint or Hermitian if

$$A^* = A$$

Theorem 0.1.0.9. *Let $\langle , \rangle$ be a Hermitian form on a complex finite dimensional vector space V. Let A be the matrix associated to $\langle , \rangle$ relative to a basis. Then A is a Hermitian Matrix and*

$$\langle X, Y \rangle = X^* AY$$

Further, if A is Hermitian then X^*AY is a Hermitian form

Proof. The proof is essentially identical to the one for real symmetric bilinear forms. The only thing which is left to check is Hermitian symmetry and I will leave it to you to check that. □

Lemma 0.1.0.10. *The real hermitian matrixes are the real symmetric matrixes*

Proof. Immediate. □

0.2 Change of Base

Change of Base for Hermitian Form

Theorem 0.2.0.11. *Let A be the matrix associated to a Hermitian form with respect to a basis B. Then the matrixes which represent the same form with respect to different basis are those of the form*

$$A' = QAQ^*$$

for some invertible matrix $Q \in GL_n(\mathbb{C})$

Proof. Let $P \in GL_n(\mathbb{C})$ be the matrix from B to B'. Let X, Y represent v, w with respect to B and X', Y' represent v, w with respect to B'. Further let A' be the matrix associated with $\langle , \rangle$ relative to the basis B'. Then we have

$$PX = X'$$

$$PY = Y'$$

and

$$X^*AY = \langle v, w \rangle = (X')^*A'Y' = (PX)^*A'(PY) = X^*(P^*A'P)Y$$

and so we have $A = P^*A'P$ or $A' = QAQ^*$ where $Q = (P^*)^{-1}$ □

Unitary Matrices

Definition 0.2.0.12. For Hermitian forms, the analog of orthogonal matrixes are called Unitary Matrixes. A matrix is unitary if

$$P^*P = I \text{ or } P^* = P^{-1}$$

For example

$$1/\sqrt{2}\begin{bmatrix} cc1 & i \\ 1 & -i \end{bmatrix}$$

is unitary.

Lemma 0.2.0.13. *The unitary matrixes for a group*

$$U_n = \{P | P^*P = I\}$$

Proof. Immediate □

Change of base and Standard Hermitian Prod

Corollary 0.2.0.14. *A change of base preserves the standard hermitian product (i.e. $X^*Y = X'^*Y'^*$) if and only if the change of base matrix is unitary.*

Proof. Immediate □

0.3 Carry over from bilinear forms

Orthogonal, Positive Definite definitions

Definition 0.3.0.15. Let $\langle , \rangle$ be a Hermitian form. We can say that two vectors v, w are orthogonal if and only if

$$\langle v, w \rangle = 0$$

Similarly we can define $\langle . \rangle$ to be positive definite if and only if

$$\langle v, v \rangle \text{ is a positive real number if } v \neq 0$$

Hermitian Space

Definition 0.3.0.16. We say a complex vector space V with a positive definite hermitian form is a Hermitian Space

Carry over from bilinear forms.

Theorem 0.3.0.17. *We then have Sylvesters theorem carries over to the case of hermitian forms.*

A Hermitian form has a orthonormal basis if and only if it is positive definitie

If $W \subseteq V$ and $\langle , \rangle$ restricted to W is non-degenerate then

$$V = W \oplus W^{\perp}$$

Proof. This proof is identical to the one for real vector spaces. □

0.4 Spectral Theorem

Notice that now we have two seemingly different interpretation of an $n \times n$ matrix. Such a matrix can be associated to

- A bilinear form $\langle , \rangle : V \times V \to F$
- A linear transformation $T : V \to V$.

Now we are going to combine these two uses of matrixes.

0.4.1 Theorems

Recall from last semester

Linear Operator change of base

Theorem 0.4.1.1. *Let $T : V \to V$ be a linear operator on V and let M be the matrix associated to it with respect to a basis $B = (v_1, \ldots, v_n)$.*

Now let $P \in GL_n(F)$ be the matrix associated with a change of base from B

to $B' = (w_1, \ldots, w_n)$. *Then the matrix associated to* T *with respect tot he basis* B' *is*

$$M' = PMP^{-1}$$

Proof. Let $v = \Sigma a_i v_i = \Sigma b_i w_i$. So in particular we know that

$$\begin{aligned} Tv \text{ expressed with basis } B' \quad &= M'(v \text{ expressed with basis })B' && (1)\\ &= P \circ M(v \text{ expressed with basis })B && (2)\\ &= P \circ MP^{-1}(v \text{ expressed with basis })B' && (3)\\ & && (4)\end{aligned}$$

□

Hermitian Operators, Unitary Operators

Theorem 0.4.1.2. *Let* $T : V \to V$ *be a linear operator on a hermitian space* V. *Further let* M *be the matrix associated to* T *with respect to an orthonormal basis. Then*

(a) The matrix M *is hermitian if and only if* $\langle v, Tw\rangle = \langle Tv, w\rangle$ *for all* $v, w \in V$. *In this case we say* T *is a* *Hermitian Operator*

(b) The matrix M *is unitary if and only if* $\langle v.w\rangle = \langle Tv, Tw\rangle$ *for all* $v, w \in V$. *In this case we say* T *is a* *Unitary Operator*

Proof. Let X, Y be the coordinate vectors so that $v = BX, w = BY$ and hence $\langle v, w\rangle = X^*Y$ and $Tv = BMX$.

Part (a):

We have $\langle v, Tw \rangle = X^*(MY)$ and $\langle Tv, w \rangle = (MX)^*Y = X^*M^*Y$.

In particular we therefore have $(\forall v, w \in V)\langle v, Tw \rangle = \langle Tv, w \rangle$ if and only if $M = M^*$ or M is hermitian. Part (b):

Similarly we have $\langle Tv, Tw \rangle = (MX)^*(MY) = X^*(M^*M)Y$

So, in particular we therefore have $(\forall v, w \in V)\langle Tv, Tw \rangle = \langle v, w \rangle$ if and only if $M^*M = I$, or M is unitary. □

Spectral Theorem

Theorem 0.4.1.3 (Spectral Theorem)**.** *(a) Let T be a hermitian operator on a hermitian vector space V Then there is an orthonormal basis for V consisting of eigenvectors of T.*

(b) Matrix form Let M be a hermitian matrix. There is a unitary matrix P such that PMP^* is a real diagonal matrix.

Proof. WE WILL PROVE THIS NEXT TIME. □

0.5 TODO

- Go through Lang's book on the same topics.