Lecture Notes Math 371: Algebra (Fall 2006)

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0.1 TALK SLOWLY AND WRITE NEATLY!!

0.2 Orthogonal

0.2.1 Definitions

Today we are going to continue with some linear algebra. Recall the definition of orthogonal

Orthogonal Vectors

Definition 0.2.1.1. Let $\langle , \rangle$ be a bilinear form on a vector space V. Then we say two vectors $v, w \in V$ are Orthogonal if

$$\langle v, w \rangle = 0$$

(this is written $v \perp w$)

Now that we know what it means for two vectors to be orthogonal we can define what it means for a vector to be orthogonal to a subspace.

Definition 0.2.1.2. Let $\langle , \rangle$ be a bilinear form on a vector space V. Let $W \subseteq V$ be a subspace. The we day that $v \in V$ is orthogonal to W if

$$(\forall w \in W)\langle v, w\rangle = 0$$

then we say that v is orthogonal to W.

Perpendicular Space

Definition 0.2.1.3. Let $\langle , \rangle$ be a bilinear form on a vector space V. Let $W \subseteq V$ be a subspace. We then define the orthogonal complement of W to be

$$W^{\perp} = \{v : (\forall w \in W)\langle v, w\rangle = 0\}$$

Null Space, Non-Degenerate

Definition 0.2.1.4. Let $\langle , \rangle$ be a bilinear form on a vector space V. We define the Null Space to be

$$N = V^{\perp} = \{v : (\forall w \in V)\langle v, w\rangle = 0\}$$

We say a bilinear form is Non-Degenerate if $V^{\perp} = \{0\}$.

0.2.2 Theorems

There is non-self orthogonal vector

Theorem 0.2.2.1. *Let $\langle , \rangle$ be a non-identically zero symmetric bilinear form on V. Then there is a $v \in V$ such that $\langle v, v \rangle \neq 0$*

Proof. To say that $\langle , \rangle$ isn't 0 means that there are $v, w \in V$ such that $\langle v, w \rangle \neq 0$. Now if $\langle v, v \rangle \neq 0$ or $\langle w, w \rangle \neq 0$ we are done.

So lets assume that $\langle v, v \rangle = \langle w, w \rangle = 0$

Then we have

$$\langle v+w, v+w \rangle = \langle v, v \rangle + \langle w, v \rangle + \langle v, w \rangle + \langle w, w \rangle = 2\langle v, w \rangle \neq 0$$

□

Vector space is product of subspace and ortho

Theorem 0.2.2.2. *Let $\langle , \rangle$ be bilinear form on a vector space V over F. Let $w \in V$ be such that $\langle w, w \rangle \neq 0$, and let $W = \{cw : c \in F\} = Span(w)$. Then*

$$V = W \oplus W^{\perp}$$

Proof. We have to show two things

(a) $W \cap W^{\perp} = 0$. This is clear because if cw is orthogonal to w then

$$\langle cw, w \rangle = c\langle w, w \rangle$$

which is 0 if and only if $c = 0$

(b) $W, W^{\perp}$ span V. I.e. every vector $v \in V$ can be written as $v = aw + v'$ where $v' \in W^{\perp}$. To show this we want to solve

$$0 = \langle v', w \rangle = \langle v - aw, w \rangle = \langle v, w \rangle - a\langle w, w \rangle$$

So if we let $a = \langle v, w \rangle / \langle w, w \rangle$ then we are good. And this is okay as we have assumed $\langle w, w \rangle \neq 0$.

□

Theorem 0.2.2.3. *Let $\langle , \rangle$ be a bilinear form on a vector space V. Let $W \subseteq V$ be a subspace of V such that $\langle , \rangle|W$ is non-degenerate. Then*

$$V = W \oplus W^{\perp}$$

Proof. ***************TODO Theorem 7.2.7 ***************MAY GIVE AS HOMEWORK □

Orthogonal basis always exist or symmetric bi

Theorem 0.2.2.4. *Let $\langle , \rangle$ be a symmetric bilinear form on a finite dimensional vector space V. Then there is an orthogonal basis for V*

Proof. Base Case:

Let $dim(V) = 1$. Now let $v \in V$. We have that either

- $\langle v, v \rangle = 0$ in which case we let $v_1 = v$ and we have

$$\langle v_1, v_1 \rangle = 0$$

- $\langle v, v\rangle > 0$ in which case we let

$$v_1 = \sqrt{\langle v, v\rangle} v$$

  and we have $\langle v_1, v_1\rangle = 1$

- $\langle v, v\rangle < 0$ in which case we let

$$v_1 = \sqrt{-\langle v, v\rangle} v$$

  and we have $\langle v_1, v_1\rangle = -1$

Inductive Case: Assume this theorem is true if $dim(V) = n-1$ and let $dim(V) = n$

Assume the form is not identically 0. Then there is a vector w such that $\langle w_1, w_1\rangle \neq 0$.

Now let $W = Span(w_1)$. Then

$$V = W \oplus W^{\perp}$$

But we know that $dim(W^{\perp}) = n - 1$ and so there is a basis $(w_2, \cdots, w_n)$ for $W^{\perp}$ such that $\langle w_i, w_j \rangle = 0$ if $i \neq j$.

But we then also have $\langle w_1, w_i \rangle = 0$ if $i > 1$ as $w_i \in W^{\perp}$ and that $(w_1, w_2, \ldots, w_n)$ is a basis for $V = W \oplus W^{\perp}$

So all that is left is to normalize the vectors. For each w_i. We have that either

- $\langle w_i, w_i \rangle = 0$ in which case we let $v_i = w_i$ and we have
$$\langle v_i, v_i \rangle = 0$$
- $\langle w_i, w_i \rangle \geq 0$ in which case we let
$$v_i = \sqrt{\langle w_i, w_i \rangle} v$$
and we have $\langle v_i, v_i \rangle = 1$

- $\langle w_i, w_i \rangle < 0$ in which case we let

$$v_i = \sqrt{-\langle w_i, w_i \rangle} v$$

and we have $\langle v_i, v_i \rangle = -1$

and so in particular we know that if $(v_1, v_2, \cdots, v_n)$ is an orthogonal basis for $W^\perp$ such that $\langle v_i, v_j \rangle = 0$ if $i \neq j$ and ± 1 or 0 if $i = j$. $\square$

Definition 0.2.2.5. Let V be a finite dimensional vector space and let $\langle , \rangle$ be a symmetric bilinear form on V. Further let $(v_1, \cdots, v_n)$ be an orthogonal basis for V with $\langle v_i, v_j \rangle = 0$ if $i \neq j$ and ± 1 if $i = j$

Then let $p = |\{i : \langle v_i, v_i \rangle = 1\}|$ and $m = |\{i : \langle v_i, v_i \rangle = -1\}|$. Then (p, m) is called the Signature of the bilinear form.

Theorem 0.2.2.6 (Sylvester's Law)**.** *Let $\langle , \rangle$ be a symmetric bilinear form on a finite dimensional vector* *space V. Then the signature (p, m) of $\langle , \rangle$ does not depend on the basis used to compute it.*

Proof. Let $r = p + m$ and let $(v_1, \ldots, v_n)$ be a basis for V of the type we are considering such that $\langle v_k, v_k \rangle = 0$ if and only if $k > r$

The first step is to show that r is independent of the form by proving that the vectors $(v_{r+1}, \ldots, v_n)$ form a basis for the null space $N = V^{\perp}$. This will show that $r = dim(v) - dim(N)$ and hence doesn't depend on the choice of basis.

A vector $w \in N$ if and only if it is orthogonal to every element of our basis. Let

$$w = c_1 v_1 + \cdots + c_n v_n$$

Since $\langle v_i, v_j \rangle = 0$ for all $i \neq j$ we see that $\langle w, v_i \rangle =$

$c_i\langle v_i, v_i\rangle$. But $\langle v_i, v_i\rangle \neq 0$ if and only if $i > r$.

And so in particular this means that $c_i = 0$ for $i \leq r$ and hence $w \in Span(v_{r+1}, \ldots, v_n)$.

Hence $V^{\perp} \subseteq Span(v_{r+1}, \ldots, v_n)$

But we also have $\langle v_k, v_k\rangle = 0$ if $k > r$ and so in fact $v_k \in N$

Therefore $N = Span(v_{r+1}, \ldots, v_n)$

Now we just need to show that p, m are determined. This is a little more complicated.
Lets assume we have two different basis $(v_1, \ldots, v_p, \ldots, v_n)$ and $(v_1^*, \ldots, v_{p^*}^*, \ldots v_n^*)$

Claim 0.2.2.7. $(v_1, \ldots, v_p, v_{p^*+1}^*, \ldots, v_n^*)$ *is linear in-*

dependent.

Proof. Assume there is a linear relation between the vectors. We may write this as

$$v_1 v_1 + \cdots + b_p v_p = c_{p^*+1} v_{p^*+1} + \cdots + c_n v_n^*$$

Let v be the vector defined by either side of this equation. Now lets compute $\langle v, v \rangle$.

$$\langle v, v \rangle = b_1^2 \langle v_1, v_1 \rangle + \cdots + b_p^2 \langle v_p, v_p \rangle = b_1^2 + \cdots + b_p^2 \geq 0$$

But we also have that

$$\langle v, v \rangle = c_{p^*+1}^2 \langle v_{p^*+1}^*, v_{p^*+1}^* \rangle + \cdots + c_n^2 \langle v_n^*, v_n^* \rangle = -c_{p^*+1}^2 - \cdots - c_{m^*}^2 \leq 0$$

So we have $\langle v, v \rangle = 0$ and so we know that $b_i = 0$ for all $i \leq p$

But then we have

$$0 = c_{p^*+1} v_{p^*+1} + \cdots + c_n v_n^*$$

and so $c_j = 0$ for all $j \geq p^* + 1$

Hence $(v_1, \dots, v_p, v^*_{p^*+1}, \dots, v^*_n)$ is linear independent.

But then we know that $|(v_1, \dots, v_p, v^*_{p^*+1}, \dots, v^*_n)| = p + n - p' \leq n$

$\square$

So $p \leq p'$ and by symmetry we therefore have $p = p'$. $\square$

Characterization of Non-Degenerate Matrixes

Theorem 0.2.2.8. *Let A be the matrix of a symmetric bilinear form with regards to some basis. Then*

(a) The null space of the form is the set of vectors v such that the coordinate vector X of v is a solution of the homogeneous equation $AX = 0$

(b) The form is non-degenerate if and only if the matrix A is non-singular

Proof. First recall the definition of a singular matrix.

Definition 0.2.2.9. We say a matrix A associated to a linear operator $T : V \to V$ is singular if any one of the following equivalent conditions on the linear operator hold

(a) $Ker(T) > 0$

(b) $Image(T) < V$

(c) $det(A) = 0$

(d) 0 is an eigenvalue of T

We say a matrix is non-singular if it isn't singular.

As we have chosen a basis we know

$$\langle X, Y \rangle = X^t A Y$$

with respect to our basis.

Part (a):

If $AY = 0$ then we have $X^t AY = 0$ for all X. And if $AY \neq 0$ then there is some coordinate (say i) in which Y is not 0. But the ith coordinate of AY is $e_i^t AY$ and so one of the products is not 0.

Part (b):

This follows from Part (a). □

0.2.3 Definitions

Orthogonal Projection

Definition 0.2.3.1. Let $\langle , \rangle$ be a bilinear form on V, and let $W \subseteq V$ such that $\langle , \rangle$ is non-degenerate on W. Then we know by the previous theorem that

$$V = W \oplus W^{\perp}$$

So in particular we know that for each $v \in V$ there is a unique $w \in W, w' \in W^{\perp}$ such that $w_v + w'_v = v$ and $\langle w_v, w'_v \rangle = 0$.

Define the orthogonal projection of V onto W to be the map $\pi : V \to W$ such that $\pi(v) = w_v$.

0.2.4 Theorem

Orthogonal Projection Characterization

Theorem 0.2.4.1. *Let $(w_1, \cdots w_r)$ be an orthonormal basis for $W \subseteq V$ and let $v \in V$. Then the orthogonal projection $\pi(v)$ of v onto W is the vector*

$$\pi(v) = \langle v, w_1\rangle w_1 + \cdots + \langle v, w_r\rangle w_r$$

Proof. Let the vector defined by the right side of the equation be w^*. Then $\langle w^*, w_i\rangle = \langle v, w_i\rangle\langle w_i, w_i\rangle = \langle v, w_i\rangle$ for $i \in \{1, \ldots, r\}$ and hence $v - w^* \in W^\perp$

But since the expression for v is unique and $V = W \oplus W^\perp$

we see that $v = w * + (v - w^*)$ is the decomposition of v into its components of $V = W \oplus W^\perp$. □

Corollary 0.2.4.2. *Let $(v_1, \cdots v_n)$ be an orthonormal basis for a Euclidean space V. Then*

$$(\forall v \in V) v = \langle v, v_1 \rangle v_1 + \cdots + \langle v, v_n \rangle v_n$$

Proof. Immediate from the previous proof □

0.3 TODO

- Go through Lang's book on the same topics.