

Lecture Notes Math 371: Algebra (Fall 2006)

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TALK SLOWLY AND WRITE NEATLY!!

0.1 Boolean Algebras

Definition of Boolean Algebra

Definition 0.1.0.1. A lattice $(B, \wedge, \vee, 0, 1)$ is a Boolean algebra if

- 1 is the greatest element and 0 is the least element.
- B is distributive
- For all $a \in B$ there is a unique $a' \in B$ such that $a \wedge a' = 0$ and $a \vee a' = 1$. We say a' is the complement of a and write $a' = \neg a$.

Basic Results of Boolean Algebra

Lemma 0.1.0.2. (a) $\neg\neg a = a$

(b) $\neg(a \wedge b) = \neg a \vee \neg b$

$$(c) \neg(a \vee b) = \neg a \wedge \neg b$$

Proof. (a) $\neg(\neg a) \wedge (\neg a) = 0$ and $\neg(\neg a) \vee (\neg a) = 1$ and $a \wedge (\neg a) = 0$ and $a \vee (\neg a) = 1$ so because $\neg a$ is unique we must have $\neg\neg a = a$.

(b)

$$\begin{aligned} (\neg a \vee \neg b) \vee (a \wedge b) &= (\neg a \vee \neg b \vee a) \wedge (\neg a \vee \neg b \vee b) \\ &= 1 \wedge 1 = 1 \end{aligned}$$

$$\begin{aligned} (\neg a \vee \neg b) \wedge (a \wedge b) &= (\neg a \wedge a \wedge b) \vee (\neg b \wedge a \wedge b) \\ &= 0 \vee 0 = 0 \end{aligned}$$

(c) Is because $\neg\neg(a \vee b) = a \vee b$ □

Ideal of Boolean Algebra

Definition 0.1.0.3. Let B be a boolean algebra. A set $I \subseteq B$ is an Ideal if

- $0 \in I$

- $(\forall x \in I)y \leq x \rightarrow y \in I$
- $(\forall x, y \in I)x \vee y \in I$

It is not hard to see that this definition corresponds to the traditional definition with $*$ as \wedge because for all x , $a \wedge x \leq x$ and if $y \leq x$ then $y \wedge x = y$. (WAVE MY HANDS REGARDING PLUS)

Definition of Symmetric Difference

Definition 0.1.0.4. Let B be a boolean algebra. Let $a \Delta b = a + b = (a \wedge \neg b) \vee (b \wedge \neg a)$. We say $a \Delta b$ is the Symmetric Difference of a and b . We also define $a/b = a \wedge \neg b$.

Example:

Give example in terms of sets.

Modding out in a Boolean Algebra

Definition 0.1.0.5. Let B be a boolean algebra and let $I \subseteq B$ be an ideal. Define $a \sim_I b$ to be $a \Delta b \in I$

Lemma 0.1.0.6. *If B is a boolean algebra and $I \subset B$ is an ideal then \sim_I is an equivalence relation.*

Proof. Reflexivity

$$x\Delta x = 0 \in I$$

Symmetry

$$x\Delta y = y\Delta x \text{ so } x \sim_I y \leftrightarrow y \sim_I x$$

Transitivity

Suppose $x\Delta y, y\Delta z \in I$. We want to show that $x\Delta y \leq x\Delta y \vee y\Delta z$.

$$\begin{aligned} x \wedge \neg z &= x \wedge \neg z \wedge 1 = (x \wedge \neg z) \wedge (y \vee \neg y) \\ &= (x \wedge \neg y \wedge \neg z) \wedge (x \wedge \neg z \wedge \neg y) \\ &\leq (y \wedge \neg z) \vee (x \wedge \neg y) \end{aligned}$$

Similarly we have $\neg x \wedge z \leq (y \wedge \neg z) \vee (x \wedge \neg y)$ and so $x\Delta y \leq x\Delta y \vee y\Delta z$. \square

Lemma 0.1.0.7. *If B is a boolean algebra and $I \subset$*

B is an ideal then \sim_I is preserved by the functions \wedge, \vee, \neg .

Proof. Let $a_1 \sim_I b_1$ and $a_2 \sim_I b_2$. We then want to show $(a_1 \vee a_2) \sim_I (b_1 \vee b_2)$, $(a_1 \wedge a_2) \sim_I (b_1 \wedge b_2)$, $\neg a_1 \sim_I \neg b_1$. We will show the first of these, the rest following in a similar manner.

$$\begin{aligned} (a_1 \vee a_2) \Delta (b_1 \vee b_2) &= a_1 / (b_1 \vee b_2) \vee a_2 / (b_1 \vee b_2) \vee b_1 / (a_1 \vee a_2) \vee b_2 / (a_1 \vee a_2) \\ &\leq a_1 / b_1 \vee a_2 / b_2 \vee b_1 / a_1 \vee b_2 / a_2 = (a_1 \Delta b_1) \vee (a_2 \Delta b_2) \end{aligned}$$

□

Definition 0.1.0.8. If B is a boolean algebra and $I \subset B$ is an ideal then we can define $B / \sim_I = \{a/I : a \in B\}$ where $a/I = \{x \in B : x \sim_I a\}$ and B / \sim_I inherits \wedge, \vee, \neg from B by the previous lemma.

Definition of Boolean Ring

Theorem 0.1.0.9. *If $(B, \wedge, \vee, \neg, 0, 1)$ is a boolean algebra then $(B, \wedge, +, 0, 1)$ is a ring. We call such a ring a Boolean Ring*

Proof. For notational convenience we will say $\wedge = *$.

It is clear that both $+, *$ are commutative and that $*$ is associative.

To see $+$ is associative notice

$$\begin{aligned} (a + b) + c &= [(a/b \vee b/a) \wedge \neg c] \vee [((a \wedge b) \vee (\neg a \wedge \neg b)) \wedge c] \\ &= [(a/b \neg c) \vee (b/a \neg c)] \vee [(a \wedge b \wedge c) \vee (\neg a \wedge \neg b \wedge c)] \end{aligned}$$

But this is symmetric in a, b, c and so $(a + b) + c = (c + b) + a$. And so by commutativity we are done.

It is also clear that

$$a + 0 = (a \wedge 1) \vee (\neg a \wedge 0) = a$$

and

$$a + a = (a \wedge \neg a) \vee (\neg a \wedge a) = 0$$

So $(B, +, 0)$ is a commutative group

So all that is left to check is distributivity.

$$\begin{aligned} (a + b) * c &= ((a \wedge \neg b) \vee (\neg a \wedge b)) \wedge c \\ &= (a \wedge \neg b \wedge c) \vee (\neg a \wedge b \wedge c) \\ a * c + b * c &= ((a \wedge c) \wedge \neg(b \wedge c)) \vee (\neg(a \wedge c) \wedge (b \wedge c)) \\ &= ((a \wedge c) \wedge (\neg b \vee \neg c)) \vee ((\neg a \vee \neg c) \wedge (b \wedge c)) \\ &= (a \wedge c \wedge \neg b) \vee (\neg a \wedge b \wedge c) \end{aligned}$$

And we are done. □

Notice that in any such ring $(B, +, *)$ we have that $x^2 = x$ for all $x \in B$.

Definition of Idempotent

Definition 0.1.0.10. Let $(B, +, *)$ be a ring. If $x \in B$ and $x^2 = x$ then we say x is idempotent. If every element

of B is idempotent then B is called a Boolean Ring

Lemma 0.1.0.11. *If $(B, +, *)$ is a boolean ring then B is abelian and of characteristic 2.*

Proof. Notice that $a+b+ab+ba = a^2+b^2+ab+ba = (a+b)^2 = a+b$. Hence $ab+ba = 0$ and $aa+aa = 2a^2 = 2a = 0$ and $a = -a$. But we also have $ab = -ba = ba$. \square

Boolean Rings are Boolean Algebras

Theorem 0.1.0.12. *Let $(B, +, *, 0, 1)$ be a boolean ring. Then if we define $\wedge = *$, $a \vee b = 1 - (1-a)*(1-b)$ and $\neg a = 1 - a$. Then $(B, \wedge, \vee, \neg, 0, 1)$ is a boolean algebra.*

Proof. Commutativity

It is clear that both $*$, $+$ are commutative.

Associativity

Let $\sigma : B \rightarrow B$ be defined by $\sigma(x) = 1 - x$. It is then clear that $\sigma^2 = 1$ and that $a \vee b = \sigma^{-1}(\sigma(a)\sigma(b))$. Hence, by associativity of multiplication we see that \vee is associative (notice that σ isn't a ring homomorphism).

$$\underline{a \vee a = a}$$

$$a \vee a = 2a - a^2 = -a^2 = a.$$

Negation

Notice that $a \vee (1 - a) = 1 - (1 - a)(1 - 1 - a) = 1 - a - a^2 = 1$ and $a \wedge (1 - a) = a * (1 - a) = a - a^2 = 0$.

Distributivity

$$(a \vee b) \wedge c = (a + b - ab)c = ac + bc - abc = ac + bc - acbc = ac \vee bc = (a \wedge c) \vee (b \wedge c)$$

Lattice conditions

$$(a \wedge b) \vee a = ab + a - a^2b = a \text{ and } (a \vee b) \wedge a =$$

$$(a + b - ab)a = a^2 - ab - a^2b = a \quad \square$$

Theorem 0.1.0.13. *Going from a boolean ring to a boolean algebra as defined above is the inverse operation of going from a boolean algebra to a boolean ring.*

Proof. Not hard but needs to be checked. □

Definition of atom

Definition 0.1.0.14. Let B be a boolean algebra. An element $a \in B$ is called an atom if $a \neq 0$ and $(\forall b \in B)b \leq a \rightarrow b = a$ or $b = 0$.

Definition of atomic, atomless Boolean Algebra

Definition 0.1.0.15. A boolean algebra B is called atomic if $(\forall b \in B)(\exists a \in B)a \leq b$ and a is an atom.

A boolean algebra B is called atomless if $(\forall b \in B)b$ is not an atom.

0.1.1 Stone Representation Theorem

Definition of Maximal Ideal

Definition 0.1.1.1. Let B be a boolean algebra. A maximal ideal of B is an ideal which is contained in no proper ideals other than itself.

Theorem 0.1.1.2. *Let B be a boolean algebra. Then an ideal I is maximal if and only if for all $x \in B$ either $x \in I$ or $\neg x \in I$.*

Proof. Assume that I is maximal and there is some x such that $x, \neg x \notin I$. But then the set

$$I_x = \{y \in B \mid (\exists a \in I)y \leq x \vee a\}$$

and

$$I_{\neg x} = \{y \in B \mid (\exists a \in I)y \leq \neg x \vee a\}$$

are ideals. Since I is maximal, $I \subset I_x, I \subset I_{\neg x}$ $x \in I_x - I$ and $\neg x \in I_{\neg x} - I$ we conclude that $B = I_x = I_{\neg x}$.

So, there exists $a, b \in I$ such that $x \vee a = 1$ and $\neg x \vee b = 1$. Hence

$$\begin{aligned} a \vee b &= a \vee b \vee 0 = (a \vee b) \vee (x \wedge \neg x) = (a \vee b \vee x) \wedge \\ &(a \vee b \vee \neg x) = 1 \wedge 1 = 1. \end{aligned}$$

However we assumed I was a proper ideal and hence

$$1 \notin I \Rightarrow \Leftarrow.$$

□

For all a exists maximal ideal without a

Theorem 0.1.1.3. *For any element $a \in B$ there is a maximal ideal which does not contain a .*

Proof. Let $J = \{I : \neg a \in I \text{ and } I \text{ is a proper ideal of } B\}$. If $I \in J$ then $a \notin I$ as I is a proper ideal. Further it is clear that $J \neq \emptyset$ $\{x : x \leq \neg a\}$ is an ideal and in J .

Hence (J, \subset) is inclusive. I.e. the union of a chain of elements is an element. So by Zorn's lemma it contains a maximal element I . We need to show I is a maximal ideal.

Assume I is not a maximal ideal. Then there exists an ideal $K \supsetneq I$. But we must therefore have $\neg a \in K$. But $K \notin J$ and so $a \in K$. Hence $1 = a \vee \neg a \in K$ and K isn't a proper ideal $\Rightarrow \Leftarrow$ So I is maximal. \square

Definition 0.1.1.4. Let B be a boolean algebra. Define $J(B) = \{J : J \text{ is a maximal ideal of } B\}$.

Stone Representation Theorem

Theorem 0.1.1.5. *Let B be a boolean algebra and define $\varphi : B \rightarrow P(J(B))$ by $\varphi(a) = \{I \in J(B) : a \notin I\}$. Then φ is an injective homomorphism of boolean algebras.*

Proof. Part (a): φ is injective.

Assume not. I.e. there exists $a, b \in B$ such that $\varphi(a) = \varphi(b)$ and $a \neq b$. Hence we must have either $a \not\leq b$ or $b \not\leq a$.

Without loss of generality we can assume that $a \not\leq b$.

So $a \wedge \neg b \neq 0$ because otherwise $a \wedge b = a$ since $a = a \wedge 1 = a \wedge (b \vee \neg b) = (a \wedge b) \vee (a \wedge \neg b)$

By the previous theorem there exists a maximal ideal I such that $a \wedge \neg b \notin I$. Hence $a \notin I$ and $\neg b \notin I$. But we then have that $b \in I$. Consequently $I \in \varphi(a)$ and $I \notin \varphi(b)$. $\Rightarrow \Leftarrow$.

Part (b): $\varphi(a \vee b) = \varphi(a) \cup \varphi(b)$

Lets first prove $\varphi(a \vee b) \subseteq \varphi(a) \cup \varphi(b)$. Let $I \in \varphi(a \vee b)$ and lets assume to get a contradiction that $a \in I$ and $b \in I$. Now because I is maximal and $a \vee b \notin I$ we have $\neg(a \vee b) \in I$. But since $a, b \in I$ we also have $a \vee b \in I$ and hence $I = B \Rightarrow \Leftarrow$. So $I \in \varphi(a) \cup \varphi(b)$ and $\varphi(a \vee b) \subseteq \varphi(a) \cup \varphi(b)$

Now lets assume $I \in \varphi(a) \cup \varphi(b)$ and $I \notin \varphi(a \vee b)$. So $a \vee b \in I$ and hence $a, b \in I$. Which means $I \notin \varphi(a)$ and $I \notin \varphi(b) \Rightarrow \Leftarrow$.

Part (c): $\varphi(a \wedge b) = \varphi(a) \wedge \varphi(b)$

Similar to Part (b).

Part (d): $J - \varphi(a) = \varphi(\neg a)$.

If $I \in J - \varphi(a)$ then $I \notin \varphi(a)$ and $a \in I$. So, since I is maximal $\neg a \notin I$ and $I \in \varphi(\neg a)$ and $J - \varphi(a) \subseteq \varphi(\neg a)$

If $I \in \varphi(\neg a)$ then $\neg a \in I$. But since I is maximal this means that $a \in I$ and hence $I \notin \varphi(a)$ or equivalently $I \in J - \varphi(a)$ and hence $\varphi(\neg a) \subseteq J - \varphi(a)$ and we are done. \square

Theorem 0.1.1.6 (Stone Theorem). *Every boolean algebra is isomorphic to some algebra of sets.*

Proof. This is immediate from the previous theorem. \square