<u>Homework 5</u>

Solutions

1)

1) Ch2 sec5 Ex 4

4) (a) Prove that for rational numbers x, y, z, w $(yw \neq 0)$, $\frac{x}{y} = \frac{z}{w}$ iff $xw = yz$

$\Rightarrow$ Let $A = \frac{x}{y}$, and $B = \frac{z}{w}$ $A = B$

The definition of division gives us.

$yA = x$, $wB = z$

multiply $z = wB$

$zyA = xwB$

since $A = B \Rightarrow$ zy must equal xw + uniqueness

$\Leftarrow$ $xw = yz$

Let $A = xw$, $B = yz$ $A = B$

$(yw \neq 0) \Rightarrow \frac{A}{w} = x$, $\frac{B}{y} = z$

multiply $z = \frac{B}{y} \Rightarrow \frac{A}{w}(z) = \frac{B}{y}(x) \Rightarrow \frac{Az}{w} = \frac{Bx}{y}$

$\Rightarrow A\left(\frac{z}{w}\right) = B\left(\frac{x}{y}\right)$ since $A = B$, and uniqueness $\Rightarrow \frac{z}{w} = \frac{x}{y}$ perfect

(b) Give a proof of $\frac{x}{y} + \frac{w}{z} = \frac{xz + wy}{yz}$ for rational x, y, z, w $(yw \neq 0)$

Let $A = \frac{x}{y}$, $B = \frac{w}{z}$, $C = \frac{xz + wy}{yz}$

by definition of division of rationals

$yA = x$ $Bz = w$ $yzC = xz + wy$

$(z)yA = (z)x$ $(y)Bz = (y)w$

$\Downarrow$ $\Downarrow$

$yzA = xz$ $yzB = wy$

2)

Ex. 2

(a) 2.5 (#5) Let $x, y, z \in \mathbb{Q}$ s.t. $z = \frac{x}{y}$.

$\frac{2}{2}$

- Assume if $x, y > 0$

  then $z = \frac{x}{y} \Leftrightarrow z = x \cdot \frac{1}{y}$ by multiplicative inverse

  we know $\frac{1}{y} > 0$ $\left(\begin{array}{l}\text{since } y\left(\frac{1}{y}\right) = 1 > 0 \\ \text{and } y \text{ \& } \frac{1}{y} \text{ are the "same sign",} \\ \text{namely } (+)\end{array}\right)$

  hence, we have z is just the product of 2 (+) rationals $(x \text{ \& } \frac{1}{y})$ (i.e fractions) $\Rightarrow z$ is positive ☑

- Assume if $x, y < 0$

  then $z = \frac{x}{y} \Leftrightarrow z = x \cdot \frac{1}{y}$ by multiplicative inverse

  we know that $\frac{1}{y} < 0$ $\left(\begin{array}{l}\text{since } y\left(\frac{1}{y}\right) = 1 > 0 \\ \text{and since } y \text{ is } (-) \text{ by assumption} \\ \left(\frac{1}{y}\right) \text{ must be } (-) \text{ as well}\end{array}\right)$

  hence, we have z is the product of 2 (-) rationals $(x \text{ \& } \frac{1}{y}) \Rightarrow z$ is positive by Thm. 3 (pg. 119) ☑

  $(-x)(-y) = xy \ \forall x, y \in \mathbb{Q}$

(b) Let $x, y, z \in \mathbb{Q}$ s.t. $z = \frac{x}{y}$

- Assume if $x > 0$, $y < 0$

  then $z = x \cdot \frac{1}{y}$ by multiplicative inverse   good.

  we know $\frac{1}{y} < 0$ $\left(\begin{array}{l}\text{since } y\left(\frac{1}{y}\right) = 1 > 0 \\ \text{and } y \text{ is } (-) \text{ by assumption and } y\left(\frac{1}{y}\right) \text{ is } (+) \\ \text{so by Thm 3 (pg 119), } \frac{1}{y} \text{ must be } (-)\end{array}\right)$

  $(-x)(-y) = +xy$

  hence, we have z is the product of a (+) rational and a (-) rational $(x, \frac{1}{y}$ respectively$) \Rightarrow z$ is negative by Thm. 3

  $(+x)(-y) = -xy$ ☑

- Assume if $x < 0$, $y > 0$,

  then similarly we have the case above, but now we have z is the product of a (-) rational and a (+) rational $(x, \frac{1}{y}$ respectively$) \Rightarrow z$ is negative by Thm.

  ☑ $(-x)(+y) = -xy$

3)

3
5.6)

2/9

a. $\left(\frac{-39}{8} \times \frac{9}{11}\right) + \left(\frac{39}{-8} \times \frac{-5}{33}\right)$

$= \left(\frac{-39}{8} \times \frac{9}{11}\right) + \left(\frac{-39}{8} \times \frac{-5}{33}\right)$ $\qquad \frac{a}{-b} = -\frac{a}{b} = \frac{-a}{b}$

$= \frac{-39}{8} \times \left(\frac{9}{11} + \frac{-5}{33}\right)$ $\qquad$ Distributive

$= \frac{-39}{8} \times \left(\frac{27}{33} + \frac{-5}{33}\right)$ $\qquad \frac{ka}{kb} = \frac{a}{b}, \; k \neq 0$

$= \frac{-39}{8} \times \frac{22}{33}$ $\qquad$ Addition

$= \frac{-13}{4} \times \frac{11}{11} = \frac{-13}{4} \times 1 = \boxed{\frac{-13}{4}}$ $\qquad$ Cancelation; $a/a = 1, a \neq 0$; Multiplicative Identity.

b. $\frac{7}{1.2} + \frac{5}{-1.8}$

$= \frac{70}{12} + \frac{50}{-18} = \frac{210}{36} + \frac{100}{-36}$ $\qquad \frac{ka}{kb} = \frac{a}{b}, \; k \neq 0$

$= \frac{210}{36} + \frac{-100}{36}$ $\qquad \frac{a}{-b} = \frac{-a}{b} = -\frac{a}{b}$

$= \frac{110}{36}$ $\qquad$ Addition

$\boxed{= \frac{55}{18}}$ $\qquad \frac{ka}{kb} = \frac{a}{b}, \; k \neq 0$

3. (2.5.6) c.

$$-6\frac{1}{4}-\frac{27}{8}\left(\frac{2}{3}-\frac{8}{9}\right)$$

$$=-\left(6\frac{1}{4}\right)-\frac{27}{8}\left(\frac{6}{9}-\frac{8}{9}\right)$$ $-a=-(a)$; $\frac{ka}{kb}=\frac{a}{b}, k\neq 0$

$$=-\left(6+\frac{1}{4}\right)-\frac{27}{8}\left(\frac{6}{9}+-\frac{8}{9}\right)$$ Definition of a mixed fraction, Definition of subtraction

$$=-\left(\frac{24}{4}+\frac{1}{4}\right)-\frac{27}{8}\left(\frac{-2}{9}\right)$$ $\frac{ka}{a}=k$; subtraction

$$=-\left(\frac{25}{4}\right)+-\left[\frac{27}{8}\left(\frac{-2}{9}\right)\right]$$ Addition, Definition of subtraction

$$=-\left(\frac{25}{4}\right)+-\left[\frac{3}{4}\left(\frac{-1}{1}\right)\right]$$ Cancelation

$$=-\left(\frac{25}{4}\right)+-\left(\frac{-3}{4}\right)$$ Multiplication

$$=\frac{-25}{4}+\frac{--3}{4}$$ $-a=-(a)$

$$=\frac{-25}{4}+\frac{3}{4}=\frac{-22}{4}=\boxed{\frac{-11}{2}}$$ $--a=a$; addition; $\frac{ka}{kb}=\frac{a}{b}, k\neq 0$

4)

2/3 2.5.7

$Ax + B = Cx + D \Rightarrow Ax + B - Cx - B = Cx + D - Cx - B$ by commutativity

$Ax - Cx + B - B = D + Cx - Cx - B$ by $x - x = 0 \Rightarrow Ax - Cx = D - B$

by dist $\Rightarrow (A-C)x = D - B$ by Thm 1 $\Rightarrow x = \frac{D-B}{A-C}$ since $A - C \neq 0$

Thus x is a rational number that is expressed by other rational numbers. good.

5)

$\frac{2}{2}$ 5. §2.5 #8

a) Let x be a nonzero rational number. Explain why $\frac{x}{0}$ cannot be defined. (Hint: Look carefully at the definition of division $\frac{x}{y}$ and see where the reasoning begins to break down if $y=0$

If $\frac{x}{0}$ existed, there would be a UNIQUE rational number y s.t.

$$0 \times y = x$$

But, if $x \neq 0$, then no such x exists according to our proof that $0 \times y = 0$.

But, if $x=0$, then $0 \times y = 0 = x$ which is true $\forall y \in \mathbb{Q}$

Hence, y is not unique and does not follow the definition.

We conclude that $\frac{x}{0}$ cannot be defined.

Notice the critical importance of uniqueness. Make sure your students understand why this is important.

b) In a) we considered the case $\frac{0}{0}$.

By definition, $\exists$ a unique y st.

$$0 \cdot y = 0$$

But, this statement is true $\forall y \in \mathbb{Q}$ and

good.

Hence, y is not unique

$\therefore$ It does not follow the definition of division and therefore $\frac{0}{0}$ is not defined.

6)

$\frac{2}{2}$ 6) Ch2 Sec6 Ex2

a) Let $x, y, z, w \in \mathbb{Q}$, and let $y, w > 0$.

Then prove that $\frac{x}{y} < \frac{z}{w} \Leftrightarrow xw < yz$

$\Longrightarrow$ $\quad y, w > 0 \Rightarrow yw > 0$

IF $\frac{x}{y} < \frac{z}{w} \Rightarrow \frac{z}{w} - \frac{x}{y} > 0$ since $yw > 0$

$yw\left(\frac{z}{w} - \frac{x}{y}\right) > yw \cdot 0 \Rightarrow \frac{ywz}{w} - \frac{ywx}{y} > 0$

$\Rightarrow yz - wx > 0 \Rightarrow yz > wx$

$\Longleftarrow$

IF $xw < yz$, since $yw > 0 \Rightarrow \frac{1}{yw} > 0$

$\frac{1}{yw}(xw) < \frac{1}{yw}(yz)$

$\Rightarrow \frac{x}{y} < \frac{z}{w}$

b) Give examples to show that both are false without the assumption that $y, w > 0$

$\frac{x}{y} < \frac{z}{w} \Rightarrow xw < yz$

$x = 1$, $y = 4$ $\quad z = -1$ $\quad w = 8$

so $-\frac{1}{4} < -\frac{1}{8}$ but $(1)(8) < (-4)(-1)$

$\Rightarrow 8 < 4$ False

$xw < yz \Rightarrow \frac{x}{y} < \frac{z}{w}$

$x = 2$

$y = -1$ So $(2)(1) < (-1)(-5) \Rightarrow 2 < 5$

$w = 1$

$z = -5$ but $\frac{2}{-1} < \frac{-5}{1} \Rightarrow -2 < -5$ False

c) $\dfrac{32.5}{-3}$, $\dfrac{-30\frac{2}{3}}{2\frac{4}{5}}$

$$-\frac{(32.5)(2\frac{4}{5})}{(\cdot 3)(2\frac{4}{5})} , -\frac{(30\frac{2}{3})(3)}{3(2\frac{4}{5})}$$

$$-\left(32+\frac{5}{10}\right)\left(2+\frac{8}{10}\right) , -\left(30+\frac{2}{3}\right)3$$

$$-\left(64+\frac{32\cdot \overset{4}{8}}{\cancel{10}\,5}+1+\frac{40}{100}\right) , -92$$

$$-\left(65+\frac{128}{5}+\frac{2}{5}\right) , -92$$

$$-\left(65+\frac{130}{5}=65+26\right),$$

$$= -91 , -92$$

Since the numbers are both negative we find a common denominator, and see which rational number has more steps to the left for that sequence given by the common denominator

$\dfrac{-30\frac{2}{3}}{2\frac{4}{5}}$ takes 92 steps to the left

and $\dfrac{32.5}{-3}$ only takes 91 steps to the left

so

$$\frac{-30\frac{2}{3}}{2\frac{4}{5}} < \frac{32.5}{-3}$$

good.

7)

(b) If $y < 0$, then let $y = -1$, and the other variables be 1. The left hand side,

$$\frac{x}{y} < \frac{z}{w}$$

is satisfied since $\frac{1}{-1} < \frac{1}{1}$, but the right hand side, $xw < yz$ is not, since $1 \cdot 1 \not< (-1) \cdot 1$.

To show the opposite implication (converse) doesn't hold either, let $w = -1$ and all other variables be 1. Then,

$$(1)(-1) < (1)(1)$$

but,

$$\frac{1}{1} \not< \frac{1}{-1}$$

(c) It's difficult to tell immediately which value is larger. Let's first get rid of the mixed fractions, and use the invert and multiply rule (p. 130) simplify them, so we have values,

$$\frac{\frac{325}{10}}{-3} \text{ and } \frac{-\frac{30 \cdot 3+2}{3}}{\frac{2 \cdot 5+4}{5}}$$

$$\frac{-325}{30} \text{ and } -\frac{460}{42}$$

Since multiplication by positive numbers preserves ordering (D), we can multiply both values of the above by $30 \cdot 42$, and compare these new numbers. To reiterate, preservation of ordering means that if our new number of the left, for example, is less than the new number of the right, then the above number of the left is less than the above number on the right. We get numbers,

$$-13650 \text{ and } -13800$$

good.

Therefore, the rightmost value is less than the leftmost value.

So whats your conclusion about the original two #'s?

2/3

7 2.6.4 absolute values

(a) Given the task of simplifying,

$$|x - 1| - 5 < \frac{2}{3}$$

into the form $|x| < z$, where z is a constant, we first add 5 (refer to HW exercise for correctness),

$$|x - 1| < \frac{17}{3}$$

Thus $|x - 1|$ lies within the interval $[0, \frac{17}{3})$, or,

$$-\frac{17}{3} < x - 1 < \frac{17}{3}$$

Adding 1 to all sides,

$$-\frac{14}{3} < (x - 1) < \frac{20}{3}$$

(x)

(b) The reduction for the next problem follows the same steps, except we use the axiom $x < y \Leftrightarrow -y < -x$ (subtract $x + y$ from both sides).

$$11 - |3 + 2x| > 2.5$$

$$-|3 + 2x| > -8.5$$

$$|3 + 2x| < 8.5$$

Using the theorem,

$$-8.5 < 3 + 2x < 8.5$$

Subtracting 3,

$$-11.5 < 2x < 5.5$$

Dividing by 2,

$$-5.75 < x < 2.75$$

(c) If we are given $|x| \geq z$, the cases when this will hold true are all values when $|x| < z$ fails. This is, therefore, $-z \not< x$ *or* $x \not< z$, i.e. $x \leq -z$ or $z \leq x$. The *or* logically corresponds to saying x could be in the region below – entirely less than – the original interval, or in the region above. Therefore,

$$|2x - \frac{3}{5}| \geq \frac{1}{5}$$

is equivalent to the statement,

$$2x - \frac{3}{5} \leq -\frac{1}{5} \text{ or } \frac{1}{5} \leq 2x - \frac{3}{5}$$

Adding $\frac{3}{5}$ to both sides [of both inequalities] yields,

$$2x \leq \frac{2}{5} \text{ or } \frac{4}{5} \leq 2x$$

Finally, dividing by 2 yields,

$$x \leq \frac{1}{5} \text{ or } \frac{2}{5} \leq x$$

(d) The expression $3 - |2x - 5| \geq 4.2$ will never be true, since $3 - |z|$ must be ≤ 3, so it cannot be ≥ 4.2.

good.

1/2

8 2.6.8 Squares implying equivalence for positive rationals

Given $x, y \in \mathbb{Q}, x, y > 0$, we want to show that $x^2 = y^2 \Leftrightarrow x = y$. $X \Leftrightarrow Y$ is broken down into cases $X \Rightarrow Y$ and $Y \Rightarrow X$ (we start with the second here because it's easier).

8)

8) a) $x^2 = y^2 \Longleftrightarrow x = y$ First $x^2 = y^2 \rightarrow x = y$ so $x^2 = y^2$

2/2

is $x^2 - y^2 = 0$ and $(x-y)(x+y) = 0$ so either $x-y$ or $x+y$ has to equal 0 so $x - y = 0$ is $x = y$ so $x^2 = y^2 \rightarrow x = y$. Now $x = y \longrightarrow x^2 = y^2$ multiply both sides by x

$x(x) = y(x)$ but $x = y$ so $x(x) = y(y)$ or $x^2 = y^2$. QED.

b) $x^2 < y^2 \longleftrightarrow x < y$ First $x^2 < y^2 \leftrightarrow x^2 - y^2 < 0$ by (B), so $x^2 - y^2 < 0 \longleftrightarrow (x-y)(x+y) < 0$, $x - y < 0 \leftrightarrow x < y$. by (B)

so $\boxed{x^2 < y^2 \longleftrightarrow x < y}$ QED

9)

$\left(\frac{9}{9}\right)$ 9. §2.6 #13

Show that for all numbers x, y and $c \neq 0$,

$$|x+y|^2 \leq (1+\tfrac{1}{c^2})|x|^2 + (1+c^2)|y|^2$$

By distribution $|x+y|^2 \leq |x|^2 + \frac{|x|^2}{c^2} + |y|^2 + |y|^2c^2 \iff$

~~Proof of Theorem 1~~ $|t|^2 = t^2$ $\quad (x+y)^2 \leq x^2 + \frac{x^2}{c^2} + y^2 + y^2c^2 \iff$

Expanding power $x^2+2xy+y^2 \leq x^2 + \frac{x^2}{c^2} + y^2 + y^2c^2 \iff$

adding $(-x^2-y^2)$ $\quad -x^2-y^2+x^2+2xy+y^2 \leq x^2 + \frac{x^2}{c^2} + y^2 + y^2c^2 - x^2 - y^2 \iff$

adding $(-2xy)$ $\quad -2xy+2xy \leq \frac{x^2}{c^2} + y^2c^2 - 2xy \iff$

commutativity $\quad 0 \leq \frac{x^2}{c^2} - 2xy + y^2c^2 \iff$

By definition $\quad 0 \leq (\frac{x}{c})^2 - 2xy + (cy)^2 \iff$

factoring $\quad 0 \leq (\frac{x}{c} - cy)^2 \iff$

Proof of theorem 1 $\quad 0 \leq |\frac{x}{c} - cy|^2$ $\quad$ good.

This statement is true since the absolut value of ANY rational number is greater than or equal to 0. The square of any positive rational number is positive (or greater than or equal to zero)

10)

~~10)~~ a) $ab \le \left(\frac{a+b}{2}\right)^2$ So multiply out the right

2/2

$ab \le \frac{a^2}{4} + \frac{ab}{2} + \frac{b^2}{4}$ Subtract so $\frac{ab}{2}$ So $\frac{ab}{2} \le \frac{a^2}{4} + \frac{b^2}{4}$

multiply by 4 so $2ab \le x^2 + y^2$ which is thm 1 since $a, b > 0$ $2(ab) \le a^2 + b^2$

If $a = b$ equality obviously holds. $ab = \left(\frac{a+b}{2}\right)^2 \longrightarrow a = b$

$ab = \frac{a^2}{4} + \quad + \frac{b^2}{4}$ so multiply through by 4 $4ab = a^2 + 2ab + b^2$

Subtract $4ab$ So $a^2 - 2ab + b^2 = 0$ Factor

$\cancel{(a+b)}(a-b)^2 = 0$ So if $a - b = 0$ then $a = b$ so proven.

b) Lets say a rectangle has length a and width b then say we have a square with side length X. So the perimeter of the rectangle is $2a + 2b$ and the perimeter of the square is $4x$. So each side of a square is $4x = 2a + 2b$ or $x = \frac{2a+2b}{4}$ So the area of the square would be $\left(\frac{a+b}{2}\right)^2$ and the area of the square is (ab) but by (a) $ab \le \left(\frac{a+b}{2}\right)^2$ So the area of the square is always larger than a rectangle with the same perimeter. Only if $a = b$ (the rectangle is a square) are their areas the same. QED