Name: Mandy PID: ______

TA: ______ Sec. No: ______ Sec. Time: ______

Math 10A.
Midterm Exam 1
December 7, 2010

Turn off and put away your cell phone.
You may use one page of notes, but no books or other assistance during this exam.
You may leave answers in symbolic form, for example $\sqrt{42}$ *or* $\ln(6)$.
Read each question carefully, and answer each question completely.
Show all of your work; no credit will be given for unsupported answers.
Write your solutions clearly and legibly; no credit will be given for illegible solutions.

If any question is not clear, ask for clarification.

#	Points	Score
1	12	
2	9	
3	6	
4	6	
5	4	
6	8	
7	6	
8	9	
9	5	
Σ	65	

1. (12 points) Suppose that f and g are functions with $f(1) = 2$, $f'(1) = 3$, $g(1) = 4$, $g'(1) = 6$. Find $h'(1)$ if:

(a) $h(x) = f(x)g(x)$

$$h'(1) = f'(1)g(1) + f(1)g'(1)$$
$$= 3 \cdot 4 + 2 \cdot 6 = 24$$

(b) $h(x) = \dfrac{\ln(x)}{f(x)}$

$$h'(x) = \frac{\frac{f(x)}{x} - \ln(x) \cdot f'(x)}{(f(x))^2}$$

$$h'(1) = \frac{2-0}{2^2} = \frac{1}{2}$$

(c) $h(x) = g(x^2)$

$$h'(x) = g'(x^2) \cdot (2x)$$
$$h'(1) = g'(1) \cdot 2 = 6 \cdot 2 = 12$$

(d) $h(x) = \arctan(f(x))$

$$h'(x) = \frac{1}{1+(f(x))^2} \cdot f'(x)$$
$$h'(1) = \frac{1}{1+2^2} \cdot 3 = \frac{1}{5} \cdot 3 = \frac{3}{5}$$

2. (9 points) Evaluate the given limit or explain why it does not exist.

(a) $\displaystyle\lim_{x\to\infty} \frac{3x^3 - 6x}{x^2 - 2x^3}$

$$\frac{3x^3-6x}{x^2-2x^3} \approx \frac{3x^3}{-2x^3} \quad \text{when } x \text{ large}$$

$$= \frac{3}{-2}$$

$$\therefore \lim_{x\to\infty} \frac{3x^3-6x}{x^2-2x^3} = -\frac{3}{2}$$

(b) $\displaystyle\lim_{x\to 0} \frac{|x|}{x}$

Does not exist.

Since

$$\lim_{x\to 0^-} \frac{|x|}{x} = \lim_{x\to 0^-} \frac{-x}{x} = \lim_{x\to 0^-} -1 = -1$$

$$\lim_{x\to 0^+} \frac{|x|}{x} = \lim_{x\to 0^-} \frac{x}{x} = \lim_{x\to 0^-} 1 = 1$$

left hand limit $\neq$ right hand limit

(c) $\displaystyle\lim_{h\to 0} \frac{\frac{1}{3+h} - \frac{1}{3}}{h}$

$$= \lim_{h\to 0} \frac{\frac{3-(3+h)}{3(3+h)}}{h} = \lim_{h\to 0} \frac{-h}{h(3(3+h))}$$

$$= \lim_{h\to 0} \frac{-1}{3(3+h)} = -\frac{1}{9}$$

3. (6 points) Suppose that $P(t)$ is the population of Whoville t years after 1990.

(a) Explain the practical meaning of the equation $P(12) = 5050$.

The population of whoville is 5050 after 12 years after 1990 (i.e 2002)

(b) Explain the practical meaning of the equation $P^{-1}(3800) = 7$.

It is 7 years after 1990 (i.e. 1997) when the population of whoville is 3800

(c) What are the units of $P'(t)$?

people / year

(d) Explain the practical meaning of the equation $P'(10) = 250$.

The instantaneous rate of change of population of whoville is 250 at 10 years after 1990 (i.e. 2000)

4. (6 points) The amount of caffeine (in milligrams) left in a person's body t hours after drinking a cup of coffee containing 100 milligrams of caffeine is given by the formula $Q(t) = 100(0.84)^t$.

(a) How many hours after drinking the coffee until only 20% of the caffeine is left in the body? (Please leave your answer in symbolic form, for example ln(6).)

$$100(0.84)^t = 20\% \cdot 100$$

$$(0.84)^t = 0.2$$

$$t \cdot \ln 0.84 = \ln 0.2$$

$$t = \frac{\ln 0.2}{\ln 0.84}$$

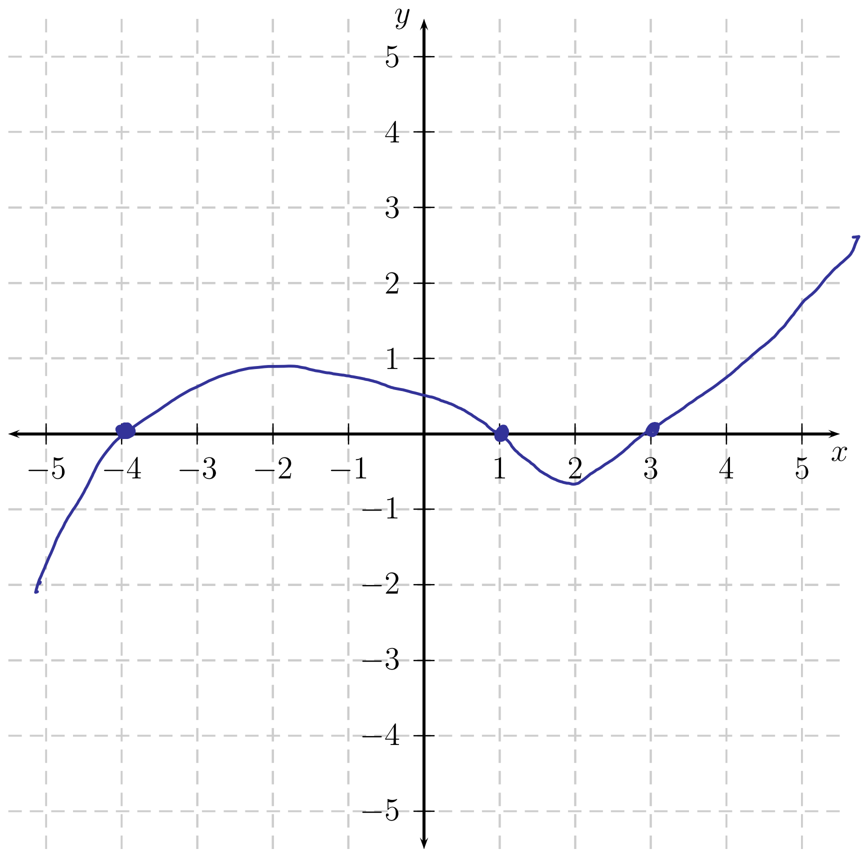
(b) Find $Q'(2)$. (Please leave your answer in symbolic form, for example ln(6).)

$$Q'(t) = 100 \cdot (0.84)^t \ln(0.84)$$

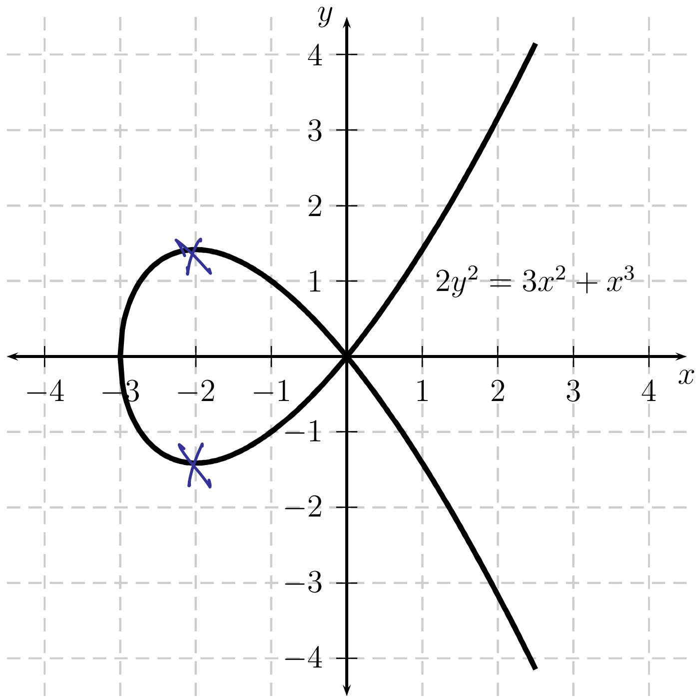
$$Q'(2) = 100 \ln 0.84 \cdot (0.84)^2$$

5. (4 points) On the axes provided below, sketch the graph of a function f with the following properties:

- $f(-4) = f(1) = f(3) = 0$
- $f'(x) > 0$ if $x < -1$ or $x > 2$
- $f'(x) < 0$ if $-1 < x < 2$
- $f''(x) < 0$ if $x < 0$
- $f''(x) > 0$ if $x > 0$



6. (8 points) The graph of the equation $2y^2 = 3x^2 + x^3$ appears below:



(a) How many points are there on the graph at which the tangent line is horizontal? Write the number of points here and also clearly mark the points on the graph.

2 points when $x = -2$
$y = 1.5$ and -1.5

(b) Use implicit differentiation to find $\frac{dy}{dx}$.

$$2y^2 = 3x^2 + x^3$$

$$2 \cdot 2y \frac{dy}{dx} = 6x + 3x^2$$

$$\frac{dy}{dx} = \frac{6x + 3x^2}{4y}$$

(c) Use your formula for $\frac{dy}{dx}$ from part (b) to find the exact (x, y) coordinates of the points from part (a).

for points in (a), $\frac{dy}{dx} = 0$

$\therefore \ 6x + 3x^2 = 0$ for $y \neq 0$

$2x + x^2 = 0$

$x(2+x) = 0$

$x = -2$ (as $x=0$, $y=0$ so we exclude $x=0$)

$2y^2 = 3x^2 + x^3$

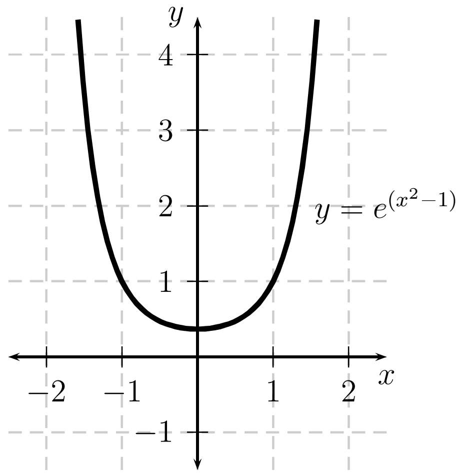
$= 3 \cdot (-2)^2 + (-2)^3 = 12 - 8 = 4$

$y^2 = 2$

$\therefore y = \pm\sqrt{2}$

$\therefore \ (x, y) = (-2, \sqrt{2})$ & $(-2, -\sqrt{2})$

7. (6 points) The graph of the function $f(x) = e^{(x^2-1)}$ appears below.



(a) Find the equation of the tangent line to the graph of f at the point with $x = -1$.

$$f'(x) = e^{x^2-1} \cdot (2x)$$

$$f'(-1) = e^0 (2) \cdot (-1) = -2$$

$$f(-1) = e^0 = 1$$

$$\therefore \quad y = -2x + b \quad \Big\} \Rightarrow \quad 1 = -2(-1) + b \quad \therefore \; b = -1$$

$$\therefore \; y = -2x - 1$$

(b) Will the tangent line approximation to $f(x)$ near $x = -1$ give an over-estimate or an under-estimate? Briefly explain your answer. (You may want to use the graph.)

under-estimate

Because the tangent line is below the graph of function

8. (9 points) Let $g(x) = x^4 - 4x^3$.

(a) Find all of the critical points of g and classify them as local minima, local maxima, or neither.

$$g'(x) = 4x^3 - 12x^2$$

$$g'(x) = 0$$

$$4x^2(x-3) = 0$$

$\therefore$ critical points $x=0,\ x=3$

$$g''(x) = 12x^2 - 24x$$

$g''(0) = 0$ $\therefore$ $x=0$ is neither local min/max

$g''(3) = 36 > 0$ $\therefore$ $x=3$ is local min

(b) Find all value(s) of x at which the graph of g has an inflection point.

$$g''(x) = 0$$

$$12x^2 - 24x = 0$$

$$12x(x-2) = 0$$

$\therefore$ inflection point : $x = 0, 2$

(c) Find the global maximum and the global minimum values of $g(x)$ (if they exist) on the interval $(0, \infty)$.

$g(0) = 0$

$g(3) = 3^4 - 4 \cdot (3)^3 = 81 - 108 = -17$

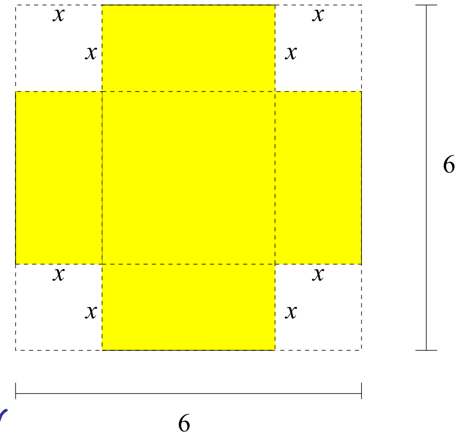
Since $g(x)$ is increasing when $x > 3$

$\therefore$ $x = 3$ is global min

As $g(x)$ is increasing without bounded when $x > 3$,

there is no global max

9. (5 points) A box with an open top is made from a square piece of material by removing equally sized squares from each corner and turning up the sides. Find the **dimensions** of the box of largest volume that can be made in this way if the square piece of material is 6 feet by 6 feet.



Volume $= V(x) = (6-2x)^2 \cdot x$

$= (36 - 24x + 4x^2) \cdot x$

$= 36x - 24x^2 + 4x^3$

$V'(x) = 36 - 48x + 12x^2$

$V'(x) = 0$

$3 - 4x + x^2 = 0$

$(x-3)(x-1) = 0$

$\Rightarrow x = 3$ or $x = 1$

if $x = 3$, $6 - 2x = 6 - 2(3) = 0$

$\therefore$ the box has no base $\Leftarrow$ impossible

$V''(x) = -48 + 24x$

$V''(1) = -48 + 24 = -24 < 0$

$\therefore x = 1$ is a max

$\therefore$ largest volume

$= (6 - 2(1))^2 \cdot 1 = 4^2 \cdot 1 = 16$