MATH 232A: Introduction to Algebraic Geometry I, Homework 1

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1. Draw the pictures of Spec($\mathbb{Z}$), Spec($\mathbb{R}$), Spec($\mathbb{C}[x]$), Spec($\mathbb{R}[x]$).

2. (Weak Nullstellensatz) Let $\Bbbk$ be any field and A a finitely generated $\Bbbk$-algebra. If A is a field then A is a finite algebraic extension of $\Bbbk$.

3. (Strong Nullstellensatz) Let $\Bbbk$ be an algebraic closed field and let $I \subseteq \Bbbk[x_1, \ldots, x_n]$ be an ideal. Then
$$\sqrt{I} = \mathbf{I}\big(\mathbf{V}(I)\big).$$

4. (Hilbert basis theorem) If A is Noetherian then $A[x]$ is Noetherian.

5. If X is a Noetherian topological space. Every nonempty closed $Y \subseteq X$ can be written as $Y = Y_1 \cup \cdots \cup Y_r$, where Y_i are irreducible and closed.
If we require $Y_i \neq Y_j$ for any $i \neq j$, then this decomposition is unique.

6. a. Let Y be the plane curve $y = x^2$ (i.e. Y is the zero set of the polynomial $f = y - x^2$). Show that $A(Y)$ is isomorphic to a polynomial ring in one variable over $\Bbbk$.
b. Let Z be the plane curve $xy = 1$. Show that $A(Z)$ is not isomorphic to a polynomial ring in one variable over $\Bbbk$.

7. (The twisted cubic curve) Let $Y \subseteq \mathbb{A}^3$ be the set $Y = \{(t, t^2, t^3) | t \in \Bbbk\}$. Show that Y is an affine variety of dimension 1. Find generators for the ideal $I(Y)$. Show that $A(Y)$ is isomorphic to a polynomial ring in one variable over $\Bbbk$. We say that Y is given by the parametric representation $x = t, y = t^2, z = t^3$.