MATH 20E Midterm 2 Winter 2013

1. Calculate the integral $\int\int\int_W \frac{e^{x^2+y^2+z^2}}{\sqrt{x^2+y^2+z^2}}\, dxdydz$ over the region W given by $1 \le x^2+y^2+z^2 \le 4$, with $z \ge 0$.

We change into spherical coordinates

$$\begin{cases} x = \rho \sin\phi \cos\theta \\ y = \rho \sin\phi \sin\theta \\ z = \rho \cos\phi \end{cases}$$

where $0 \le \theta \le 2\pi$, $0 \le \phi \le \frac{\pi}{2}$, $1 \le \rho \le 2$ (3)

& Jacobian $= \rho^2 \sin\phi$ (2)

$$\therefore \iiint_W \frac{e^{x^2+y^2+z^2}}{\sqrt{x^2+y^2+z^2}}\, dxdydz$$

$$= \int_0^{2\pi}\int_0^{\pi/2}\int_1^2 \frac{e^{\rho^2}}{\rho}\rho^2 \sin\phi \, d\rho\, d\theta\, d\phi \quad (1)$$

$$= 2\pi^{(1)} \left(-\cos\phi\right)_0^{\pi/2 \,(1)} \cdot \frac{1}{2}\int_1^2 e^{\rho^2 \,(1)} d\rho^2$$

$$= 2\pi \cdot \frac{1}{2} e^{\rho^2}\Big|_1^2$$

$$= \pi(e^4 - e) \quad (1)$$

2. Let c be the part of the parabola $y = x^2$ running from the point $(-1, 1)$ to the point $(1, 1)$.

(a) Calculate $\int_c \mathbf{F} \cdot d\mathbf{s}$ for the vector field $\mathbf{F}(x, y, z) = (x^2y, xz, y)$.

(b) Would you expect to get the same value for the integral as in (a) for *any* curve from $(-1, 1)$ to $(1, 1)$? Why, or why not?

a) $c(t) = (t, t^2, 0) \quad t \in [-1, 1]$ (2)

$$\int_c F \cdot ds = \int_{-1}^{1} (t^4, 0, t^2) \cdot (1, 2t, 0)\, dt$$ (2) (2)

$$= \int_{-1}^{1} t^4\, dt = \frac{2}{5}$$ (1)

b) No (1)

Since $\nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2y & xz & y \end{vmatrix} = (1-x, 0, z-x^2) \neq 0$ (1)

So F cannot be ∇f for any function. (1)

Hence the integral would depend on the choice of path

3. Calculate the surface area of the part of the cylinder $x^2 + y^2 = 1$ for which $0 \leq z \leq 3 + 2x$

The parametrization is $\Phi(\theta, z) = (\cos\theta, \sin\theta, z)$

$0 \leq \theta \leq 2\pi, \quad 0 \leq z \leq 3 + 2\cos\theta$ ③

$\Phi_\theta = (-\sin\theta, \cos\theta, 0)$ ②

$\Phi_z = (0, 0, 1)$

$\Rightarrow \Phi_\theta \times \Phi_z = (\cos\theta, \sin\theta, 0)$ ②

$\|\Phi_\theta \times \Phi_z\| = 1$

$\therefore$ Surface area $= \iint_S 1 \, dS$

$= \int_0^{2\pi} \int_0^{3+2\cos\theta} 1 \, dz \, d\theta$ ②

$= \int_0^{2\pi} (3 + 2\cos\theta) \, d\theta$

$= 6\pi$ ①

4. Calculate the flow of the vector field $\mathbf{F}(x, y, z) = (x, y, z)$ through the part of the hyperboloid $z^2 = 1 + x^2 + y^2$ with $1 \le z \le \sqrt{2}$, with the normal vector pointing to the upper side of this surface.

Only need to set up integral

Method 1: $\Phi(r,\theta) = (r\cos\theta, r\sin\theta, \sqrt{1+r^2})$ $0 \le r \le 1$, $0 \le \theta \le 2\pi$ (3)

$\Phi_r = \left(\cos\theta, \sin\theta, \frac{r}{\sqrt{1+r^2}}\right)$ (2)

$\Phi_\theta = (-r\sin\theta, r\cos\theta, 0)$

$\Phi_r \times \Phi_\theta = \left(\frac{-r^2\cos\theta}{\sqrt{1+r^2}}, \frac{-r^2\sin\theta}{\sqrt{1+r^2}}, r\right)$ (2)

↑ pointing to upper side

(2)

$$\therefore \iint_S F \cdot d\vec{S} = \int_0^{2\pi}\int_0^1 (r\cos\theta, r\sin\theta, \sqrt{1+r^2}) \cdot \left(\frac{-r^2\cos\theta}{\sqrt{1+r^2}}, \frac{-r^2\sin\theta}{\sqrt{1+r^2}}, r\right) dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^1 \left(\frac{-r^3}{\sqrt{1+r^2}} + r\sqrt{1+r^2}\right) dr\,d\theta \quad (1)$$

$$= \int_0^{2\pi}\int_0^1 \frac{r}{\sqrt{1+r^2}}\, dr\,d\theta$$

Method 2: $\Phi(x,y) = (x, y, \sqrt{1+x^2+y^2})$ (2)

$\Phi_x = \left(1, 0, \frac{x}{\sqrt{1+x^2+y^2}}\right)$ (2)

$\Phi_y = \left(0, 1, \frac{y}{\sqrt{1+x^2+y^2}}\right)$

$\Phi_x \times \Phi_y = \left(\frac{-x}{\sqrt{1+x^2+y^2}}, \frac{-y}{\sqrt{1+x^2+y^2}}, 1\right)$ (2)

$$\iint_S F \cdot d\vec{S}$$

$$= \iint_D (x, y, \sqrt{1+x^2+y^2}) \cdot \left(\frac{-x}{\sqrt{1+x^2+y^2}}, \frac{-y}{\sqrt{1+x^2+y^2}}, 1\right) dA$$

$$= \iint_D \left(\frac{-x^2}{\sqrt{1+x^2+y^2}} + \frac{-y^2}{\sqrt{1+x^2+y^2}} + \sqrt{1+x^2+y^2}\right) dA \quad (2)$$

$$= \iint_D \frac{1}{\sqrt{1+x^2+y^2}}\, dA$$

By using polar coordinate $x = r\cos\theta$, $y = r\sin\theta$

↙ (1) for Jacobian

$$\iint_S F \cdot dS = \int_0^{2\pi}\int_0^1 \frac{1}{\sqrt{1+r^2}} \cdot r\, dr\,d\theta \quad (1)$$