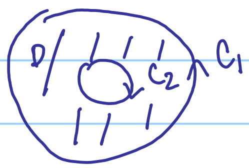
20E week 9

§8.1

Q9 Verify Green's Theorem for $\int_C (2x^3-y^3)\,dx+(x^3+y^3)\,dy$

$D:\ a\le x^2+y^2\le b$

Sol:



Green's thm:

$$\int_C (2x^3-y^3)\,dx+(x^3+y^3)\,dy=\iint_D \left(3x^2-(-3y^2)\right)dx\,dy$$

RHS: $D.\ \Phi(r,\theta)=(r\cos\theta, r\sin\theta)\quad 0\le\theta\le 2\pi,\ a\le r\le b$

$$\text{RHS}=\int_a^b\int_0^{2\pi}\left(3r^2\cos^2\theta+3r^2\sin^2\theta\right)r\,d\theta\,dr$$

$$=\int_a^b\int_0^{2\pi}3r^3\,d\theta\,dr=2\pi\int_a^b 3r^3\,dr$$

$$=2\pi\left[\frac{3r^4}{4}\right]_a^b=\frac{3\pi}{2}(b^4-a^4)$$

$$\text{LHS}=\int_C (2x^3-y^3)\,dx+(x^3+y^3)\,dy$$

$$=\int_{C_1}(2x^3-y^3)\,dx+(x^3+y^3)\,dy+\int_{C_2}(2x^3-y^3)\,dx+(x^3+y^3)\,dy$$

$c_1(\theta)=(b\cos\theta, b\sin\theta),\quad c_2(\theta)=(a\cos\theta, a\sin\theta)$

$$=\int_0^{2\pi}\left[(2b^3\cos^3\theta-b^3\sin^3\theta)(-b\sin\theta)+(b^3\cos^3\theta+b^3\sin^3\theta)(b\cos\theta)\right]d\theta$$
$$+\int_{2\pi}^{0}\left[(2a^3\cos^3\theta-a^3\sin^3\theta)(-a\sin\theta)+(a^3\cos^3\theta+a^3\sin^3\theta)(a\cos\theta)\right]d\theta$$

Note $\int_0^{2\pi}\left[(2b^3\cos^3\theta-b^3\sin^3\theta)(-b\sin\theta)+(b^3\cos^3\theta+b^3\sin^3\theta)(b\cos\theta)\right]d\theta$

$$=b^4\int_0^{2\pi}\left(-2\cos^3\theta\sin\theta+\sin^4\theta+\cos^4\theta+\sin^3\theta\cos\theta\right)d\theta$$

$$=b^4\left(+2\int_0^{2\pi}\cos^3\theta\,d(\cos\theta)+\int_0^{2\pi}(1-\cos^2\theta)^2+\cos^4\theta\,d\theta+\int_0^{2\pi}\sin^3\theta\,d\sin\theta\right)$$

$$=b^4\left(\cancel{2\cdot\left[\frac{\cos^4\theta}{4}\right]_0^{2\pi}}+\int_0^{2\pi}(1-2\cos^2\theta)\,d\theta+\cancel{\left[\frac{\sin^4\theta}{4}\right]_0^{2\pi}}\right)$$

$$=b^4\left(2\pi+\int_0^{2\pi}\frac{1+\cos 2\theta}{2}\,d\theta\right)=b^4\left(2\pi-\frac{1}{2}\left[2\pi+\frac{\sin 2\theta}{2}\right]_0^{2\pi}\right)=\frac{3\pi}{2}b^4$$

$\therefore\ \text{LHS}=\frac{3\pi}{2}(b^4-a^4)=\text{RHS}$

$\therefore$ Vertify green's thm

Q12 let $P(x,y) = \frac{-y}{x^2+y^2}$, $Q(x,y) = \frac{x}{x^2+y^2}$.
Assuming D is the unit disk, investigate why Green's thm fail.

Sol. Note $(0,0) \in D$

But $P(0,0)$, $Q(0,0)$ is not defined.

So we don't have $P, Q : D \to \mathbb{R}$ are of class C^1

$\therefore$ Condition for Green's thm fail

So Green's thm cannot work!