# 20E Week 8

## § 7.5

Q2 Evaluate $\iint_S xyz\, dS$

where $S$ is the $\triangle$ with vertices $(1,0,0)$, $(0,2,0)$, $(0,1,1)$

Sol Need to find the plane that contains the triangle.

many ways. Here is one of those

Let the plane containing $S$ be $ax+by+cz=d$

By substituting the vertices, we get the eqn :

$$2x+y+z=2.$$

$\Rightarrow$ Unit normal $\vec{n} = \frac{1}{\sqrt{6}}(2\vec{i}+\vec{j}+\vec{k})$

The domain $D$ in $xy$-plane is the triangle with vertices $(1,0)$, $(0,2)$ & $(0,1)$

Now $dS = \frac{dx\,dy}{\vec{n}\cdot\vec{k}} = \sqrt{6}\, dx\,dy$

$$\iint_S f\, dS = \iint_D xy(2-2x-y)\cdot\sqrt{6}\, dx\,dy$$

$$= \sqrt{6}\int_0^1\int_{1-x}^{2(1-x)} [2(x-x^2)y - xy^2]\, dy\,dx$$

$$= \frac{\sqrt{6}}{30}$$

Q4 Evaluate $\iint_S (x+y+z)\, dS$

where $S$ is the boundary of the unit ball $B$

Sol:

$$\iint_S (x+y+z)\, dS = \iint_{S_1}(x+y+z)\, dS + \iint_{S_2}(x+y+z)\, dS$$

where $S_1$ is upper hemisphere & $S_2$ is lower hemisphere.

Note $(x,y,z)$, $(-x,-y,-z)$ are in opp. hemisphere

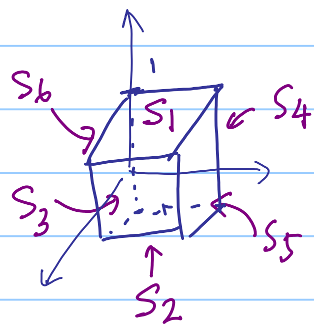
& $x+y+z = -(x+y+z)$

$\therefore \iint_{S_2}(x+y+z)\, dS = -\iint_{S_1}(x+y+z)\, dS$

$\therefore \iint_S (x+y+z)\, dS = 0$

8) Evaluate $\iint_S z^2 dS$ where $S$ is bdry $C = [-1,1]\times[-1,1]\times[-1,1]$

Sol:




$S_1$: for $z=1$

we have $x \in [-1,1]$, $y \in [-1,1]$

$T_x \times T_y = \vec{i} \times \vec{j} = \vec{k}$

$\Rightarrow \|T_x \times T_y\| = 1$

$\therefore \iint_{S_1} z^2 dS = \int_{-1}^{1}\int_{-1}^{1} 1\cdot 1 \, dx\,dy$

$= 4$

Similarly $\iint_{S_2} z^2 dS = 4$

For $S_3$: $\iint_{S_3} z^2 dS = \int_{-1}^{1}\int_{-1}^{1} z^2 \, dy\,dz = \frac{4}{3}$

Similarly, $\iint_{S_4} z^2 dS = \iint_{S_5} z^2 dS = \iint_{S_6} z^2 dS = \frac{4}{3}$

$\therefore \iint_S z^2 dS = 4+4+4\cdot\left(\frac{4}{3}\right) = \frac{40}{3}$

Q10 $S$ is $z = \sqrt{R^2 - x^2 - y^2}$ where $0 \le x^2 + y^2 \le R^2$.
The mass density at $(x,y,z) \in S$ is given by $m(x,y,z) = x^2+y^2$.
Find total mass of $S$.

Sol. Need to evaluate $\iint_S (x^2+y^2)\,dS$

let $\Phi(r,\theta) = \left(r\cos\theta, r\sin\theta, \sqrt{R^2-r^2}\right)$, $0 \le \theta \le 2\pi$, $0 \le r \le R$

$\therefore \Phi_r = \left(\cos\theta, \sin\theta, \frac{-r}{\sqrt{R^2-r^2}}\right)$

$\Phi_\theta = (-r\sin\theta, r\cos\theta, 0)$

$\Phi_r \times \Phi_\theta = \left(\frac{r^2\cos\theta}{\sqrt{R^2-r^2}}, \frac{r^2\sin\theta}{\sqrt{R^2-r^2}}, r\right)$

$\|\Phi_r \times \Phi_\theta\| = \sqrt{\frac{r^4}{R^2-r^2} + r^2} = \sqrt{\frac{R^2r^2}{R^2-r^2}} = \frac{Rr}{\sqrt{R^2-r^2}}$

$$\therefore \iint_S (x^2+y^2)\,dS = \int_0^R \int_0^{2\pi} r^2 \cdot \frac{Rr}{\sqrt{R^2-r^2}}\,d\theta\,dr$$
$$= 2\pi R \int_0^R \frac{r^3}{\sqrt{R^2-r^2}}\,dr$$

let $r = R\sin\phi$

$$dr = R\cos\phi\,d\phi, \quad \sqrt{R^2-r^2} = R\cos\phi$$
$$= 2\pi R \int_0^{\pi/2} \frac{R^3\sin^3\phi\,d\phi}{R\cos\phi} \cdot R\cos\phi\,d\phi$$
$$= 2\pi R^4 \int_0^{\pi/2} \sin^3\phi\,d\phi$$
$$= 2\pi R^4 \int_0^{\pi/2} (1-\cos^2\phi)\sin\phi\,d\phi$$

$$u = \cos\phi, \quad du = -\sin\phi\,d\phi$$
$$= 2\pi R^4 \int_1^0 -(1-u^2)\,du$$
$$= 2\pi R^4 \left(-u + \frac{u^3}{3}\right)_1^0$$
$$= 2\pi R^4\left(1-\frac{1}{3}\right) = \frac{4\pi R^4}{3}$$

## 7.6

**Q3** let $S$ be the closed surface that consists of the hemisphere $x^2+y^2+z^2=1, z\ge 0$ and its base $x^2+y^2\le 1$, $z=0$.
let $E$ be the electric defined by $E(x,y,z) = 2x\vec{i} + 2y\vec{j} + 2z\vec{k}$.
Find the electric flux across $S$.

Sol: Write $S = H \cup D$, $H$: upper hemisphere & $D$: base

$$\therefore \iint_S E\cdot dS = \iint_H E\cdot dS + \iint_D E\cdot dS$$

$\boxed{H}$ The unit normal $\vec{n}$ pointing outward from $H$ is $x\vec{i}+y\vec{j}+z\vec{k}$.

$$\therefore \iint_H E\cdot dS = \iint_H (2x,2y,2z)\cdot(x,y,z)\,dS$$
$$= 2\iint_H (x^2+y^2+z^2)\,dS$$
$$= 2\iint_H 1\,dS = 4\pi$$

$\boxed{D}$ unit normal $= -k$ & $z=0$ on $D$

$$\iint_D E\cdot dS = \iint_D (2x,2y,0)\cdot(0,0,-1)\,dS = 0$$

$$\therefore \iint_S E\cdot dS = 4\pi$$

Q7 Calculate $\iint_S F\cdot dS$, where $S$ is the surface of $x^2+y^2+z^2\le 1, z\ge 0$
& $F=(x+3y^5)\vec{i}+(y+10xz)\vec{j}+(z-xy)\vec{k}$

Sol. Again we have $S=H\cup D$ as in Q3

$\boxed{H}$ In spherical coord: $\vec{r}=(\sin\phi\cos\theta)\vec{i}+(\sin\phi\sin\theta)\vec{j}+(\cos\phi)\vec{k}$

$d\vec{S}=\vec{r}\sin\phi\, dS$ $\qquad 0\le\theta\le 2\pi,\ 0\le\phi\le\frac{\pi}{2}$

$$\therefore \iint_H F\cdot dS$$
$$=\iint_H (\sin\phi\cos\theta+3\sin^5\phi\sin^5\theta,\ \sin\phi\sin\theta+10\sin\phi\cos\theta\cos\phi,\ \cos\phi-\sin^2\phi\sin\theta\cos\theta)\cdot\vec{r}\,dS$$
$$=\int_0^{\pi/2}\int_0^{2\pi}(1+3\sin^6\phi\cos\theta\sin^5\theta+9\sin^2\phi\cos\theta\cos\phi\sin\theta)\sin\phi\,d\theta\,d\phi$$
$$=2\pi$$

$\boxed{D}$ $z=0,\ x^2+y^2\le 1$, $\vec{n}=-\vec{k}$

$$\therefore \iint_D F\cdot d\vec{S}=\iint_D xy\,dx\,dy=0$$

$$\therefore \iint_S \vec{F}\cdot d\vec{S}=2\pi$$

Q10 Evaluate $\iint_S \vec{F}\cdot\vec{n}\,dA$, where $\vec{F}(x,y,z)=\vec{i}+\vec{j}+z(x^2+y^2)^2\vec{k}$
and $S$ is the surface of cylinder $x^2+y^2\le 1,\ 0\le z\le 1$

Sol $\vec{r}=(\cos\theta,\sin\theta,z)$ $\qquad 0\le\theta\le 2\pi,\ 0\le z\le 1$

unit normal is $\vec{n}=(\cos\theta,\sin\theta,0)$

$$\therefore \iint_S \vec{F}\cdot\vec{n}\,dA=\iint_S(1,1,z)\cdot(\cos\theta,\sin\theta,0)\,dA$$
$$=\int_0^{2\pi}\int_0^1(\cos\theta+\sin\theta)\,dz\,d\theta$$
$$=\int_0^{2\pi}(\cos\theta+\sin\theta)\,d\theta$$
$$=\left[\sin\theta-\cos\theta\right]_0^{2\pi}=0$$