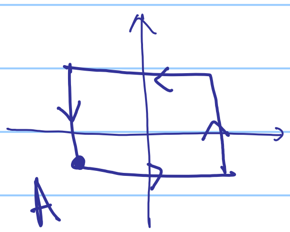
MATH 20E Week 7

§ 7.2

Q12 let $\vec{F} = (z^3 + 2xy)\vec{i} + x^2\vec{j} + 3xz^2\vec{k}$
Show integral of F around unit square with vertex $(\pm 1, \pm 1)$
is zero

Sol

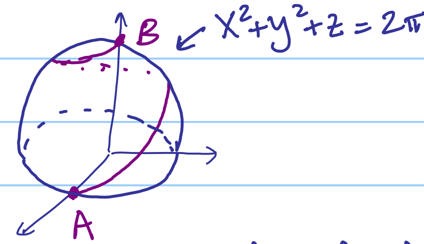


let $c(t)$ be a curve going from A
along the square to A

Note if $f(x,y,z) = xz^3 + x^2y$
Then $F = \nabla f$

$$\therefore \int_c \vec{F} \cdot ds = f(A) - f(A) = 0$$

Q 18 A cyclist rides up a mountain :



She exerts a force described by $\vec{F}(x,y,z)=y\vec{i}+x\vec{j}+\vec{k}$
What is the work done by the cyclist from A to B?
What is unrealistic about this model of a cyclist?

Sol: Note $\vec{F}=\nabla f$ where $f(x,y,z)=xy+z$
And $A=(\sqrt{2\pi},0,0)$, $B=(0,0,2\pi)$
$\therefore$ $WD=\int_C F\cdot ds = f(B)-f(A)=2\pi$

Unrealistic : any reasonable ans.
One possible ans: The path is $c(t)=\left(\sqrt{(2\pi-t)}\cos t, \sqrt{(2\pi-t)}\sin t, t\right)$
for $0\le t\le 2\pi$.
Near $t=2\pi$, $\|c'(t)\|\to\infty$ which is impossible.

7.3

Q12 Find a parametrization of $x^3+3xy+z^2=2$, $z>0$, and use it to find the tangent plane at $x=1, y=\frac{1}{3}, z=0$.
Compare your answer with that using level set.

Sol: As $z\ge 0$, $z=\sqrt{2-x^3-3xy}$
So the parametrization is
$$\Phi(u,v)=(u,v,\sqrt{2-u^3-3uv})$$
At $(1,\frac{1}{3},0)$, as $x\ne 0$, we can have
$$\Phi(u,v)=\left(u, \frac{2-u^3-v^2}{3u}, v\right)$$
$$\Phi_u=\left(1, -\frac{2-v^2}{3u^2}-\frac{2u}{3}, 0\right), \quad \Phi_v=\left(0,-\frac{2v}{3u},1\right)$$
$$\Phi_u(1,0)=\left(1,-\frac{4}{3},0\right), \quad \Phi_v(1,0)=(0,0,1)$$

$$\Phi_u \times \Phi_v = \begin{vmatrix} i & j & k \\ 1 & -4/3 & 0 \\ 0 & 0 & 1 \end{vmatrix} = \left(-\frac{4}{3}, -1, 0\right)$$

$\therefore$ tangent: $\quad -\frac{4}{3}(x-1) - 1\left(y - \frac{1}{3}\right) = 0$

level set. $\quad f(x,y,z) = x^3 + 3xy + z^2$

$$\nabla f = (3x^2 + 3y, 3x, 2z)$$

$$\nabla f\left(1, \tfrac{1}{3}, 0\right) = (4, 3, 0)$$

$\therefore$ tangent plane: $\quad (4, 3, 0) \cdot \left(x-1, y-\frac{1}{3}, z\right) = 0$

$\therefore \quad 4(x-1) + 3\left(y - \frac{1}{3}\right) = 0$

Q14 Given a sphere of radius 2 centered at the origin.
Find Tangent plane at $(1, 1, \sqrt{2})$. by Considering:

a) $\Phi(\theta, \phi) = (2\cos\theta\sin\phi, 2\sin\theta\sin\phi, 2\cos\phi)$

b) A level surface of $f(x,y,z) = x^2 + y^2 + z^2$,

c) A graph of $g(x,y) = \sqrt{4 - x^2 - y^2}$

a)
$$\Phi_\theta = (-2\sin\theta\sin\phi, 2\cos\theta\sin\phi, 0)$$
$$\Phi_\phi = (2\cos\theta\cos\phi, 2\sin\theta\cos\phi, -2\sin\phi)$$

and $\Phi\left(\frac{\pi}{4}, \frac{\pi}{4}\right) = (1, 1, \sqrt{2})$

$\therefore \Phi_\theta\left(\frac{\pi}{4}, \frac{\pi}{4}\right) = (-1, 1, 0)$, $\Phi_\phi\left(\frac{\pi}{4}, \frac{\pi}{4}\right) = (1, 1, -\sqrt{2})$

$$\Phi_\theta \times \Phi_\phi = \begin{vmatrix} i & j & k \\ -1 & 1 & 0 \\ 1 & 1 & -\sqrt{2} \end{vmatrix} = (-\sqrt{2}, -\sqrt{2}, -2)$$

$\therefore$ tangent plane: $\quad (-\sqrt{2}, -\sqrt{2}, -2) \cdot (x-1, y-1, z-\sqrt{2}) = 0$

$$2(x-1) + 2(y-1) + 2\sqrt{2}(z - \sqrt{2}) = 0$$

b)
$$\nabla f = (2x, 2y, 2z)$$
$$\nabla f(1, 1, \sqrt{2}) = (2, 2, 2\sqrt{2})$$

$\therefore$ tangent plane: $\quad 2(x-1) + 2(y-1) + 2\sqrt{2}(z - \sqrt{2}) = 0$

c)
$$\frac{\partial g}{\partial x} = \frac{1}{2}\frac{-2x}{\sqrt{4-x^2-y^2}} \qquad \frac{\partial g}{\partial y} = \frac{1}{2}\frac{-2y}{\sqrt{4-x^2-y^2}}$$

$$\frac{\partial g}{\partial x}(1,1) = -\frac{1}{\sqrt{2}} \qquad \frac{\partial g}{\partial y}(1,1) = -\frac{1}{\sqrt{2}}$$

tangent: $\quad z = \sqrt{2} - \frac{1}{\sqrt{2}}(x-1) - \frac{1}{\sqrt{2}}(y-1)$

7.4

Q4 Torus can be represented by

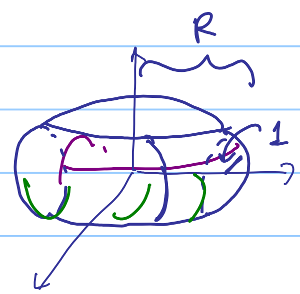
$$\Phi(\theta,\phi) = \big((R+\cos\phi)\cos\theta, (R+\cos\phi)\sin\theta, \sin\phi\big)$$

$0 \le \theta \le 2\pi,\ 0 \le \phi \le 2\pi.\ R > 1$

Show $A(T) = (2\pi)^2 R$ by using (3), (6)

Sol:
$$\frac{\partial \Phi}{\partial \theta} = \big(-(R+\cos\phi)\sin\theta,\ (R+\cos\phi)\cos\theta,\ 0\big)$$
$$\frac{\partial \Phi}{\partial \phi} = \big(-\sin\phi\cos\theta, -\sin\phi\sin\theta, \cos\phi\big)$$
$$\therefore \frac{\partial(x,y)}{\partial(\theta,\phi)} = -(R+\cos\phi)\sin\phi\ ,\quad \frac{\partial(y,z)}{\partial(\theta,\phi)} = (R+\cos\phi)\cos\theta\cos\phi$$
$$\frac{\partial(x,z)}{\partial(\theta,\phi)} = -(R+\cos\phi)\sin\theta\cos\phi$$
$$(3) \Rightarrow\quad A(T) = \iint_D \sqrt{\left(\frac{\partial(x,y)}{\partial(\theta,\phi)}\right)^2 + \left(\frac{\partial(y,z)}{\partial(\theta,\phi)}\right)^2 + \left(\frac{\partial(x,z)}{\partial(\theta,\phi)}\right)^2}\, d\theta\, d\phi$$
$$= \int_0^{2\pi}\int_0^{2\pi} \sqrt{\left[(R+\cos\phi)^2\sin^2\phi + (R+\cos\phi)^2\cos^2\phi\right]}\, d\theta\, d\phi$$
$$= 2\pi \int_0^{2\pi} (R+\cos\phi)\, d\phi = (2\pi)^2 R$$

(6):



Surface area = upper + lower

For purple one: $y = \sqrt{1-(x-R)^2} = f(x)$

$$f'(x) = \frac{-(x-R)}{\sqrt{1-(x-R)^2}}$$

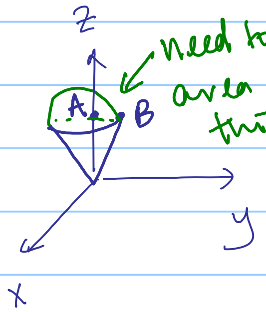
$$A(\text{upper}) = 2\pi \int_{R-1}^{R+1} |x| \cdot \sqrt{1+(f'(x))^2}\, dx$$
$$= 2\pi \int_{R-1}^{R+1} x \sqrt{1 + \frac{(x-R)^2}{1-(x-R)^2}}\, dx \qquad (\text{since } x > 0)$$
$$= 2\pi \int_{R-1}^{R+1} x \sqrt{\frac{1}{1-(x-R)^2}}\, dx$$

$x - R = \cos\theta$
$dx = -\sin\theta\, d\theta$
$1-(x-R)^2 = 1-\cos^2\theta = \sin^2\theta$

$$= 2\pi \int_\pi^0 (R+\cos\theta) \cdot \frac{1}{\sin\theta}(-\sin\theta)\, d\theta$$
$$= 2\pi\ R \cdot \pi = 2(\pi)^2 R = A(\text{lower})$$
$$\therefore A(T) = (2\pi)^2 R$$

Q6 Find the area of the portion of unit sphere cut out by the cone $z \geq \sqrt{x^2+y^2}$

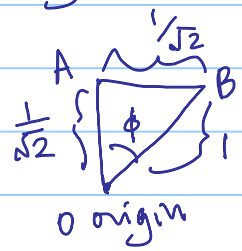
Sol: The intersection of $\begin{cases} z=\sqrt{x^2+y^2} & \text{cone} \\ x^2+y^2+z^2=1 & \text{sphere} \end{cases}$



need to find area of this

is $x^2+y^2=\frac{1}{2}$ & $z=\frac{1}{\sqrt{2}}$

Note



$\therefore \phi = \frac{\pi}{4}$

$\therefore$ parametrization is $(\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi)$

$0 \leq \theta \leq 2\pi, \quad 0 \leq \phi \leq \frac{\pi}{4}$

$$\therefore \text{Area} = \int_0^{2\pi}\int_0^{\pi/4} \sin\phi \, d\phi \, d\theta$$

$$= (2-\sqrt{2})\pi$$

Q11 Find the area of surface obtained by rotating $y=x^2$, $0 \leq x \leq 1$ about y-axis

Sol: $y=f(x)=x^2$

$f'(x)=2x$

$$\therefore A = 2\pi\int_0^1 |x| \sqrt{1+(f'(x))^2} \, dx$$

$$= 2\pi\int_0^1 x\sqrt{1+4x^2} \, dx \qquad \begin{cases} \text{let } u=1+4x^2 \\ du=8x\,dx \end{cases}$$

$$= 2\pi\int_1^5 \frac{1}{8}\cdot\sqrt{u} \, du = 2\pi\left[\frac{1}{8}\cdot\frac{2}{3}u^{3/2}\right]_1^5 = \frac{\pi}{6}\left(5^{3/2}-1\right)$$