MATH 20E - Week 6 . Winter 2013

6.1 Q2 Define $T(x^*, y^*) = \left(\frac{x^* - y^*}{\sqrt{2}}, \frac{x^* + y^*}{\sqrt{2}} \right)$

Show T rotates the unit square $D^* = [0,1] \times [0,1]$

Sol

$$T(x^*, y^* = \underbrace{\begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}}_{\overset{\shortparallel}{A}} \begin{pmatrix} x^* \\ y^* \end{pmatrix}$$

$\det A \neq 1$. By thm 1, it transforms parallelgrams to parallelgrams, vertices to vertices

Check $(1,0) \rightsquigarrow \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right)$

$(1,1) \rightsquigarrow (0, \sqrt{2})$

$(0,1) \rightsquigarrow \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right)$

So it rotates the square

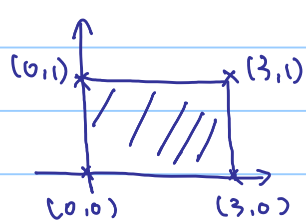
Q3 $D^* = [0,1] \times [0,1]$. $T(u,v) = (-u^2 + 4u, v)$

Find image of D. Is T 1-1 ?

Sol:

Hint for figuring out the image: Check vertex to get a sense of it

$\therefore$ Image of T



T is 1-1 because if $T(u,v) = T(u',v')$

$(-u^2 + 4u, v) = (-(u')^2 + 4u', v')$

$\Rightarrow v = v'$ & $-u^2 + 4u = -(u')^2 + 4u'$

$\Rightarrow (u')^2 - u^2 = 4(u' - u)$

$\Rightarrow (u' - u)(u' + u) = 4(u' - u)$

$\Rightarrow (u' + u - 4)(u' - u) = 0$

Then we have $u' + u - 4 = 0$ or $u' - u = 0$

But it is impossible for $u' + u - 4 = 0$ as $u', u \leq 1$

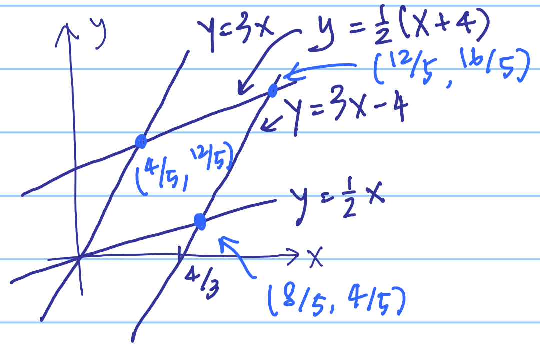
$\Rightarrow u' = u$

Now we have $u = u'$ & $v = v'$

So T is 1-1.

Q4 Let D^* be the paralogram bounded by $y = 3x - 4$, $y = 3x$, $y = \frac{1}{2}x$ and $y = \frac{1}{2}(x+4)$. let $D = [0,1] \times [0,1]$.

Find a T s.t. $T(D^*) = D$



Let $T\begin{pmatrix} x^* \\ y^* \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x^* \\ y^* \end{pmatrix}$

WANT $T(D^*) = D$

$(0,0) \rightsquigarrow (0,0)$, $\left(\frac{8}{5}, \frac{4}{5}\right) \rightsquigarrow (1,0)$, $\left(\frac{12}{5}, \frac{16}{5}\right) \rightsquigarrow (1,1)$, $\left(\frac{4}{5}, \frac{12}{5}\right) \rightsquigarrow (0,1)$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 8/5 \\ 4/5 \end{pmatrix} = \begin{pmatrix} 8/5\, a + 4/5\, b \\ 8/5\, c + 4/5\, c \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \cdots$$

By solving get $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 3/4 & -1/4 \\ -1/4 & 1/2 \end{pmatrix}$

$\therefore T(x^*, y^*) = \left(\frac{3}{4}x^* - \frac{1}{4}y^*, -\frac{1}{4}x^* + \frac{1}{2}y^*\right)$

Q10 Suppose $T: \mathbb{R}^2 \to \mathbb{R}^2$ is linear and is given by $Tx = Ax$
Show that if $\det A \neq 0$, then T takes parallelograms onto parallelograms.

Sol By hint, the general parallelogram in $\mathbb{R}^2$ can be described by

$$q = p + \lambda v + \mu w \quad \text{for } \lambda, \mu \in (0,1), \; v \text{ not scalar mult. of } w$$

$$T(q) \stackrel{\text{def}}{=} Aq$$
$$= Ap + \lambda Av + \mu Aw$$

Now Ap, Av, Aw are vectors in $\mathbb{R}^2$
and as $\det A \neq 0$, Av is not scalar mult. of Aw
$\therefore$ Image of T is parallelogram of $\mathbb{R}^2$

6.2

Q29 Let E be the ellipsoid $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 \leq 1$, $a, b, c \geq 0$
a) Find the vol. of E
b) Evaluate $\iiint_E \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 \, dx\,dy\,dz$

a) Consider

$$(x, y, z) = (a\rho \sin\phi \cos\theta, \, b\rho \sin\phi \sin\theta, \, c\rho \cos\phi)$$

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 = \rho^2 \sin^2\phi \cos^2\theta + \rho^2 \sin^2\phi \sin^2\theta + \rho^2 \cos^2\phi = \rho^2$$

So $0 \leq \rho \leq 1$

$$\frac{\partial(x, y, z)}{\partial(\rho, \theta, \phi)} = abc\,\rho^2 \sin\phi$$

$$\therefore \text{Vol of } E = \iiint_E 1 \, dx\,dy\,dz$$

$$= \int_0^1 \int_0^\pi \int_0^{2\pi} abc\,\rho^2 \sin\phi \; d\theta\, d\phi\, d\rho$$

$$= abc \cdot \left(\int_0^1 \rho^2 d\rho\right) \cdot \left(\int_0^\pi \sin\phi \, d\phi\right)\left(\int_0^{2\pi} 1 \, d\theta\right)$$

$$= abc \cdot \frac{1}{3} \cdot (2) \cdot (2\pi)$$

$$= \frac{4\pi abc}{3}$$

b) $\iiint_E \left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2+\left(\frac{z}{c}\right)^2 dx\,dy\,dz$

$$= \int_0^1\int_0^\pi\int_0^{2\pi} \rho^2\cdot\left(abc\,\rho^2\sin\phi\right) d\theta\, d\phi\, d\rho$$

$$= abc\cdot\left(\int_0^1 \rho^4 d\rho\right)\cdot\left(\int_0^\pi \sin\phi\, d\phi\right)\cdot\int_0^{2\pi} 1\, d\theta$$

$$= abc\cdot\left(\frac{1}{5}\right)\cdot(2)\cdot 2\pi = \frac{4\pi abc}{5}$$

7.2

3) Consider $F(x,y,z) = x\vec{i}+y\vec{j}+z\vec{k}$.
Compute the work done in moving along $y=x^2, z=0$ from $x=-1$ to 2

Sol. Parametrize the curve as $(t, t^2, 0)$

$$ds = (1, 2t, 0)$$

$\therefore$ Work done $= \int_C F\cdot ds$

$$= \int_{-1}^{2} (t,t^2,0)\cdot(1,2t,0)\,dt$$

$$= \int_{-1}^{2} t+2t^3\,dt$$

$$= \left(\frac{t^2}{2}+\frac{t^4}{2}\right)_{-1}^{2}$$

$$= \frac{4}{2}+\frac{16}{2}-\frac{1}{2}-\frac{1}{2} = 9$$