MATH 20E Week 4

§3.2 Taylor's Theorem

* For $f: U \subseteq \mathbb{R}^n \to \mathbb{R}$ of class C^3,

the second-order Taylor Theorem states that

$$f(x_0+h) = f(x_0) + \sum_{i=1}^{n} h_i \frac{\partial f}{\partial x_i}(x_0) + \frac{1}{2}\sum_{i,j} h_i h_j \frac{\partial^2 f}{\partial x_i \partial x_j}(x_0) + R_2(x_0,h)$$

where $\dfrac{R_2(x_0,h)}{\|h\|^2} \to 0$ as $h \to 0$

Q6 $\quad f(x,y) = e^{(x-1)^2}\cos y$ where $x_0=1,\ y_0=0$

Sol:

$$f_x = 2(x-1)e^{(x-1)^2}\cos y$$

$$f_y = -e^{(x-1)^2}\sin y$$

$$f_{xx} = 2e^{(x-1)^2}\cos y + 4(x-1)^2 e^{(x-1)^2}\cos y$$

$$f_{xy} = -2(x-1)e^{(x-1)^2}\sin y = f_{yx}$$

$$f_{yy} = -e^{(x-1)^2}\cos y$$

@ $(1,0)$

$$f(1,0)=1$$

$$f_x(1,0) = f_y(1,0) = 0$$

$$f_{xx}(1,0) = 2$$

$$f_{xy}(1,0) = f_{yx}(1,0) = 0$$

$$f_{yy}(1,0) = -1$$

So, the second order Taylor formula is

$$f(h) = 1 + h_1^2 - \frac{1}{2}h_2^2 + R_2((1,0),h) \quad \text{where } h = (h_1, h_2)$$

and where $\frac{R_2((1,0),h)}{\|h\|} \to 0$ as $\|h\| \to 0$

$\blacksquare$

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§4.2 Arc Length

Q10 $$s(t) = \int_a^t \|c'(\tau)\| \, d\tau$$

Find $s(t)$ for the curves $\alpha(t) = (\cosh t, \sinh t, t)$

$\beta(t) = (\cos t, \sin t, t)$

with $a = 0$

Sol: $\boxed{\alpha}$

$$\alpha'(t) = (\sinh t, \cosh t, 1)$$

$$\|\alpha'(t)\| = \sqrt{\sinh^2 t + \cosh^2 t + 1}$$

$$= \sqrt{2\cosh^2 t - 1 + 1} = \sqrt{2} \cdot \cosh t$$

$$\therefore s(t) = \int_0^t \|\alpha'(\tau)\| \, d\tau$$

$$= \int_0^t \sqrt{2} \cosh(\tau) \, d\tau$$

$$= \sqrt{2} \sinh(\tau) \Big|_0^t$$

$$= \sqrt{2} \sinh(\tau)$$

$\boxed{\beta}$ $$\beta'(t) = (-\sin t, \cos t, 1)$$

$$\|\beta'(t)\| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}$$

$$s(t) = \int_0^t \sqrt{2}\, d\tau = \sqrt{2}\, t$$

Q8 Let $c(t) = (t, t\sin t, t\cos t)$

Find arc length of c between $(0,0,0)$ & $(\pi, 0, -\pi)$

Sol. Note: $c(0) = (0,0,0)$

$$c(\pi) = (\pi, 0, -\pi)$$

$$c'(t) = (1, \sin t + t\cos t, \cos t, t\sin t)$$

$$\|c'(t)\| = \sqrt{1 + (\sin t + t\cos t)^2 + (\cos t + t\sin t)^2}$$

$$= \sqrt{2 + t^2}$$

So arc length $L = \int_0^{\pi} \sqrt{2+t^2}\, dt$

Here, use: $\int \sqrt{x^2+a^2}\, dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\log\left|x + \sqrt{x^2+a^2}\right|$

Now $a = \sqrt{2}$

Then $L = \frac{t}{2}\sqrt{t^2+2} + \frac{2}{2}\log\left|t + \sqrt{t^2+2}\right| \Big|_0^{\pi}$

$$= \frac{1}{2}\left[\pi\sqrt{\pi^2+2} + 2\log\left(\pi + \sqrt{\pi^2+2}\right)\right] - \frac{1}{2}\log 2$$

§4.3 Vector Fields

Q11 Sketch a few flow lines of the vector field

$$F(x,y)=(x,x^2)$$

Sol: Note: flow line of F is a path $c(t)$ s.t.

$$c'(t)=F(c(t))$$

So now let $c(t)=(x(t),y(t))$

we need to solve

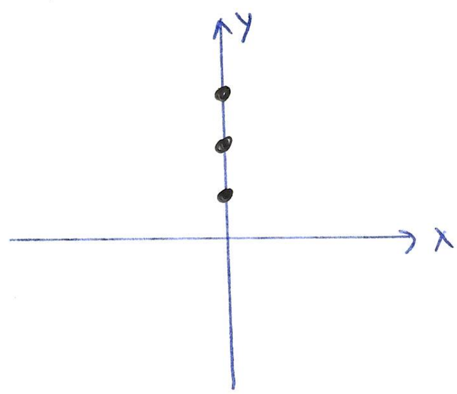
$$x'(t)=x(t) \quad \cdots ①$$

$$\& \quad y'(t)=x^2(t) \quad \cdots ②$$

① $\Rightarrow$ $x(t)=C_1e^t$ $\qquad$ C_1, C_2 const.

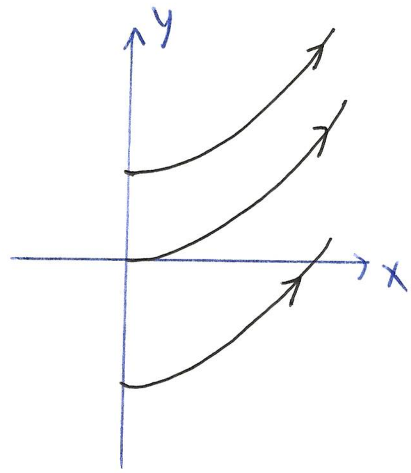
② $\Rightarrow$ $y(t)=\frac{1}{2}C_1^2e^{2t}+C_2$

Case a: if $C_1=0$, $x(t)=0$ & $y(t)=C_2$



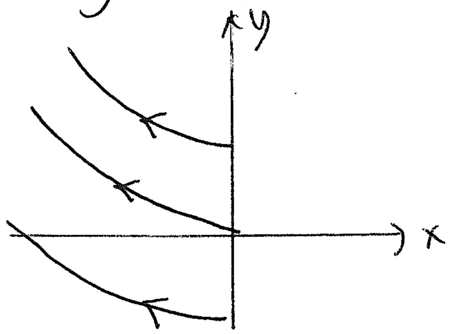
Case b: if $C_1>0 \Rightarrow x(t)>0$

and $y(t)=\frac{1}{2}x(t)^2+C_2$ $\Rightarrow$



Case C: if $c_1 < 0$, then $x(t) < 0$

and $y(t) = \frac{1}{2}(x(t))^2 + c_2$



Q14 Show $c(t) = (t^2, 2t-1, \sqrt{t})$, $t > 0$

is a flow line of $F(x,y,z) = (y+1, 2, \frac{1}{2z})$

Sol. Need to check

$$c'(t) = F(c(t))$$

$$LHS = c'(t) = (2t, 2, \frac{1}{2\sqrt{t}})$$

$$RHS = F(c(t)) = F(t^2, 2t-1, \sqrt{t})$$

$$= ((2t-1)+1, 2, \frac{1}{2\sqrt{t}})$$

$$= (2t, 2, \frac{1}{2\sqrt{t}})$$

$$= LHS$$

§4.4 Divergence and Curl

Q26 Show $F = (x^2+y^2)\vec{i} - 2xy\vec{j}$ is NOT a gradient field.

Sol:

Method 1: Suppose F is gradient field of some u

i.e. $F = \nabla u$

$$\Rightarrow \quad \frac{\partial u}{\partial x} = x^2+y^2$$

$$\frac{\partial u}{\partial y} = -2xy$$

Then $\dfrac{\partial^2 u}{\partial y \partial x} = 2y$

$\neq$ not equal

$\dfrac{\partial^2 u}{\partial x \partial y} = -2y$

This contradiction proves F is not a gradient field

Method 2:

If F is a gradient field, then $\nabla \times F = 0$.

But
$$\nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2+y^2 & -2xy & 0 \end{vmatrix} = (-2y-2y)\vec{k}$$

$$= -4y\vec{k} \neq 0$$

So F is NOT a gradient field