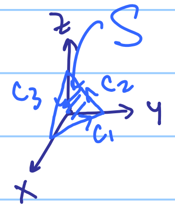
MATH20E Week 10

§8.2

Q6



let S be the triangle bounded by c.

$\Phi(s,t) = (s, t, 1-s-t)$ $\quad 0 \le s \le 1$

$0 \le t \le 1-s$

C consists of C_1, C_2, C_3 where

$C_1(t) = (1-t, t, 0) \quad 0 \le t \le 1$

$C_2(t) = (0, 1-t, t) \quad 0 \le t \le 1$

$C_3(t) = (t, 0, 1-t) \quad 0 \le t \le 1$

By Stokes thm, $\iint_S (\nabla \times F)\cdot ds = \int_C F \cdot ds$

LHS: $\nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & xz & xy \end{vmatrix} = (0,0,0)$

$\therefore$ LHS $= 0$

$$\text{RHS} = \int_C F\cdot ds = \int_{C_1} F\cdot ds + \int_{C_2} F\cdot ds + \int_{C_3} F\cdot ds$$

$$= \int_0^1 (0,0,t(1-t))\cdot(-1,1,0)\,dt + 0 + 0 = 0$$

So LHS = RHS. Stokes' thm verified!

Q10 Note the ellipsoid S is a closed surface and has no boundary

$\therefore$ By Stokes' thm, $\iint_S (\nabla\times F)\cdot d\vec{S} = \int_{\partial S} F \cdot ds = 0$

Q11 Note ∂S is the unit circle lying on x-y plane.

So we can parametrize ∂S as $(x,y,z) = (\cos\theta, \sin\theta, 0)$ $\quad 0 \le \theta \le 2\pi$

So by Stokes' thm,

$$\iint_S (\nabla\times F)\cdot \vec{n}\, dA = \int_{\partial S} \vec{F}\cdot ds$$

$$= \int_0^{2\pi} (\sin\theta, -\cos\theta, 0)\cdot(-\sin\theta, \cos\theta, 0)\, d\theta$$

$$= -2\pi$$

§8.3

Q4 We need to do Q3 above

3) If $F = \nabla f$, then $\frac{\partial f}{\partial x} = 2xyz + \sin x$ (1)

$\frac{\partial f}{\partial y} = x^2 z$ (2)

$\frac{\partial f}{\partial z} = x^2 y$ (3)

Integrate (1) with respect to x

$f = x^2yz - \cos x + h(y,z)$ where h is a fcn of y & z

Integrate (2) with respect to y

$f = x^2yz + g(x,z)$ where g is a fcn of x & z

Integrate (3) with respect to z

$f = x^2yz + k(x,y)$ where k is a fcn of x & y.

$\therefore g(x,z) = k(x,y) = -\cos x + C$ where C is a constant

$h(y,z) = C$

$\therefore f(x,y\,z) = x^2yz - \cos x + C$

So $\int_c F \cdot ds = f(c(\pi)) - f(c(0))$

$= f((-1,0,\pi/4)) - f((1,0,0))$

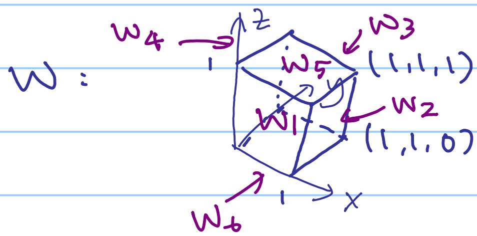
$= (-\cos(-1) + C) - (-\cos(1) + C)$

$= \cos 1 - \cos(-1)$

§8.4

Q3

$W:$



(i) $\iint_{W_1} F\cdot dS = \int_0^1\int_0^1 (x,0,z)\cdot(0,-1,0)\,dx\,dz = 0$

$\iint_{W_2} F\cdot dS = \int_0^1\int_0^1 (1,y,z)\cdot(1,0,0)\,dy\,dz = \int_0^1\int_0^1 1\,dy\,dz = 1$

$\iint_{W_3} F\cdot dS = \int_0^1\int_0^1 (x,1,z)\cdot(0,1,0)\,dx\,dz = 1$

$\iint_{W_4} F\cdot dS = \int_0^1\int_0^1 (0,y,z)\cdot(-1,0,0)\,dy\,dz = 0$

$\iint_{W_5} F\cdot dS = \int_0^1\int_0^1 (x,y,1)\cdot(0,0,1)\,dx\,dy = 1$

$\iint_{W_6} F\cdot dS = \int_0^1\int_0^1 (x,y,0)\cdot(0,0,-1)\,dx\,dy = 0$

$\therefore \iint_{\partial W} F\cdot dS = 3$

(ii) By divergence thm,

$\iint_{\partial W} F\cdot dS = \iiint_W (\nabla\cdot F)\,dV = \iiint_W (1+1+1)\,dV = 3\cdot(\text{vol of unit cube})$
$= 3$

Q5a) Parametrize W as $\Phi(r,\theta,z) = (r\cos\theta, r\sin\theta, z)$ $\quad 0\le z\le 1$, $0\le r\le\sqrt{z}$, $0\le\theta\le 2\pi$

$\nabla\cdot F = 0+0+x = x$

By Gauss' Theorem,

$$\iint_{\partial W} F\cdot dS = \iiint_W (\nabla\cdot F)\,dV$$
$$= \int_0^{2\pi}\int_0^1\int_0^{\sqrt{z}} r\cos\theta\cdot(r)\,dr\,dz\,d\theta$$
$$= (\sin\theta)\Big|_0^{2\pi}\cdot\int_0^1\int_0^{\sqrt{z}} r^2\cos\theta\,dr\,dz$$
$$= 0$$

8) Parametrize the unit ball, B, as

$$\Phi(\rho,\theta,\phi) = (\rho\cos\theta\sin\phi,\ \rho\sin\theta\sin\phi,\ \rho\cos\phi) \qquad \begin{array}{l} 0\le\theta\le 2\pi \\ 0\le\phi\le\pi \\ 0\le\rho\le 1\end{array}$$

$$\nabla\cdot F = 3y^2+3x^2+3z^2 = 3(x^2+y^2+z^2) = 3\rho^2$$

By Gauss thm,

$$\iint_S F\cdot dS = \iiint_B (\nabla\cdot F)\,dV$$
$$= \int_0^{2\pi}\int_0^{\pi}\int_0^1 3\rho^2\cdot(\rho^2\sin\phi)\,d\rho\,d\phi\,d\theta$$
$$= 2\pi\cdot(-\cos\theta)\Big|_0^{\pi}\left(\frac{3\rho^5}{5}\right)\Big|_0^1$$
$$= 2\pi\cdot 2\cdot\frac{3}{5} = \frac{12\pi}{5}$$

Practice Final Fall 03

Q5 Parametrize the region W bounded by S as

$$\Phi(r,\theta,z) = (r\cos\theta,\ r\sin\theta,\ z) \qquad \begin{array}{l} 0\le r\le 2 \\ 0\le z\le 2 \\ 0\le\theta\le\pi\end{array}$$

$$\nabla\cdot F = 1+2z$$

By Gauss thm,

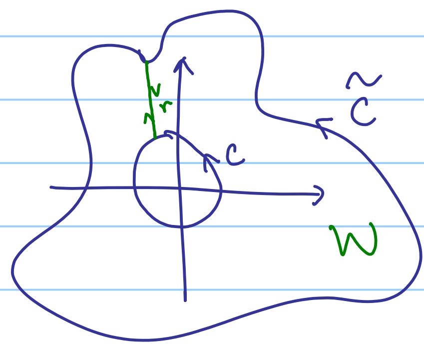
$$\iint_S F\cdot dS = \iiint_W (\nabla\cdot F)\,dV$$
$$= \int_0^{\pi}\int_0^2\int_0^2 (1+2z)\cdot r\,dr\,dz\,d\theta$$
$$= \pi\int_0^2\left[(1+2z)\cdot\frac{r^2}{2}\right]_0^2 dz$$
$$= \pi\int_0^2 2(1+2z)\,dz$$
$$= 2\pi(z+z^2)\Big|_0^2 = 2\pi(2+4) = 12\pi$$

6) a) let $c(t) = (\cos\theta, \sin\theta) \qquad 0\le\theta\le 2\pi$

$$\int_c F\cdot dr = \int_0^{2\pi}(-\sin\theta,\cos\theta)\cdot(-\sin\theta,\cos\theta)\,d\theta$$
$$= \int_0^{2\pi}(\sin^2\theta+\cos^2\theta)\,d\theta = \int_0^{2\pi} 1\,d\theta = 2\pi$$

b) The circle bound the unit disk which include (0,0)
But F is not defined at (0,0)

c)



let $\alpha(t)$ now be the curve going $\tilde{C}$ counterclockwise then going r to C (clockwise) then going (−r) to get back $\tilde{C}$

By Green's thm $\int_\alpha F \cdot ds = \iint_W \nabla \cdot F \, dV$

$$\nabla \cdot F = \frac{y(2x)}{x^2+y^2} + \frac{-x(2y)}{x^2+y^2} = 0$$

So $\int_\alpha F \cdot ds = 0$

$$\Rightarrow \int_{\tilde{C}} F \cdot ds + \left(-\int_C F \cdot ds\right) = 0$$

$$\int_{\tilde{C}} F \cdot ds = \int_C F \cdot ds = 2\pi$$