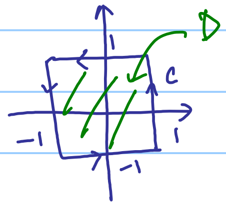
Hw 9 Sol.

便箋標題 24/11/2011

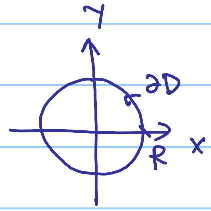
8.1

1) By Green's thm,



$$\int_C y\,dx - x\,dy = \iint_D -2\,dx\,dy$$
$$= -2\cdot(\text{Area of square})$$
$$= -8$$

2) By Green's thm, the area of a region is



$$A = \frac{1}{2}\int_{\partial D}(x\,dy - y\,dx)$$
$$= \frac{1}{2}\int_0^{2\pi}\big((R\cos\theta)(R\cos\theta) - R\sin\theta(-R\sin\theta)\big)\,d\theta$$
$$= \frac{1}{2}\int_0^{2\pi} R^2\,d\theta$$
$$= \pi R^2$$

Parametrize ∂D as
$(x,y) = (R\cos\theta, R\sin\theta)$ $\quad 0 \le \theta < 2\pi$

5) From the given parametrization of the cycloid,

$$dx = a(1-\cos\theta)\,d\theta, \qquad dy = a\sin\theta\,d\theta$$

Note the x-axis forms part of the boundary. This part of the boundary can be described by $x = 2\pi a(1-u)$, $y=0$ for $0 \le u \le 1$.
Note that our cycloid is traversed clockwise, i.e. from left to right, starting at the origin along the cycloid, and then back along the x-axis from right to left. Since $y = dy = 0$, the area is

$$A = \frac{1}{2}\int_{\partial D}(x\,dy - y\,dx)$$
$$= \frac{1}{2}\int_0^{2\pi}\big[a(\theta - \sin\theta)\cdot a\sin\theta\,d\theta - a(1-\cos\theta)\cdot a(1-\cos\theta)\,d\theta$$
$$+ \frac{1}{2}\int_0^1 (x\cdot 0 - 0\cdot dx)$$
$$= -3a^2\pi$$

7) Let D denote the unit disk, then $\partial D = C$.

By Green's thm,

$$\int_C (2x^3 - y^3)\,dx + (x^3+y^3)\,dy = \iint_D 3x^2 + 3y^2\,dA$$
$$= \int_0^{2\pi}\int_0^1 3\cdot r^2\cdot r\,dr\,d\theta$$
$$= \frac{3\pi}{2}$$

11) Parametrize D: $(x,y)=(r\cos\theta, r\sin\theta)$ $\quad 0\le r\le 1,\ 0\le\theta\le 2\pi$

$$\nabla\cdot F=2$$

$$\therefore \iint_D \nabla\cdot F\,dA=\iint_D 2\,dA=\int_0^{2\pi}\int_0^1 2r\,dr\,d\theta=2\pi$$

On the other hand, outward unit normal to unit circle is $\vec{n}=(\cos\theta,\sin\theta)$

$$\therefore \int_{\partial D} F\cdot n\,ds=\int_0^{2\pi}(\cos\theta,\sin\theta)\cdot(\cos\theta,\sin\theta)\,d\theta=2\pi$$

12) Note $P(x,y)=\frac{-y}{x^2+y^2}$, $Q(x,y)=\frac{x}{x^2+y^2}$ both are not well-defined at $(0,0)$ i.e. P,Q are not of class C^1, so it does not satisfy the condition of Green's theorem. Hence the formula in Green's theorem does not hold

15) Parametrize ellipse as $x=a\cos\theta$, $y=b\sin\theta$. $\quad 0\le\theta\le 2\pi$

By Green's thm,

$$\begin{aligned}A&=\frac{1}{2}\int_{\partial D}(x\,dy-y\,dx)\\&=\frac{1}{2}\int_0^{2\pi}\big((a\cos\theta)(b\cos\theta)-(b\sin\theta)(-a\sin\theta)\big)\,d\theta\\&=\frac{1}{2}\int_0^{2\pi}ab\,d\theta\\&=ab\pi.\end{aligned}$$

19) To form one loop of the rose, we go from $\theta=0$ to $\theta=\frac{\pi}{2}$.

By exercise 16, the area is

$$\begin{aligned}\frac{1}{2}\int_0^{\frac{\pi}{2}}(3\sin 2\theta)^2\,d\theta&=\frac{9}{2}\int_0^{2\pi}\sin^2 2\theta\,d\theta\\&=\frac{9}{2}\int_0^{2\pi}\frac{1+\cos 4\theta}{2}\,d\theta \quad \text{double angle formula}\\&=\frac{9\pi}{8}\end{aligned}$$

8.2

3) Let H be the upper hemisphere

(i) Since $\vec{F}(x,y,z)=(x,y,z)$, $\nabla\times F=0$

So $\iint_H (\nabla\times F)\cdot dS=0$

(ii) Note that the tangent to ∂H at the point $(x,y,0)$ is the vector $(-y,x,0)$ which is perpendicular to $\vec{F}=(x,y,z)$. So

$$\int_{\partial H} F\cdot ds=0.$$

So, Stokes' thm is verified.

5) Since the boundary of the surface is a closed curve, by Stokes' thm, we have

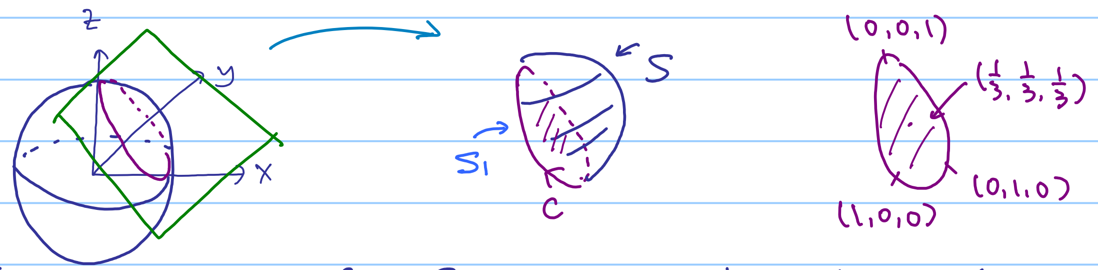
$$\iint_S (\nabla\times F)\cdot dS=\int_{\partial S} F\cdot ds$$

The boundary is the circle $x^2+y^2=1$, $z=0$.

We can then have parametrization as $x=\cos\theta$, $y=\sin\theta$, $z=0$, $0\le\theta<2\pi$.

$$\therefore \int_{\partial S} F\cdot ds=\int_0^{2\pi}(\cos\theta,\sin\theta,0)\cdot(-\sin\theta,\cos\theta,0)\,d\theta=0$$

7)



First note the surface S is the dome-shaped like surface with bdry C and we have the disk S_1 on the plane $x+y+z=1$ share the same bdry

By Stokes' thm,

$$\iint_S(\nabla\times F)\cdot dS=\int_{\partial S}F\cdot ds=\int_{\partial S_1}F\cdot ds=\iint_{S_1}(\nabla\times F)\cdot ds$$

By computation, $\nabla\times F=(-2,-2,-2)$

$$\therefore \iint_S(\nabla\times F)\cdot dS=\iint_{S_1}(\nabla\times F)\cdot\vec{n}\,ds$$
$$=\frac{-6}{\sqrt{3}}\iint_{S_1}1\,ds$$
$$=-\frac{4\pi}{\sqrt{3}}$$

$\vec{n}$ is normal of the disk which is $\frac{1}{\sqrt{3}}(1,1,1)$

(Since $\iint_{S_1}1\,ds=$ Area of S_1 $=\pi\left(\sqrt{\frac{2}{3}}\right)^2$ as centre of S_1 is $(\frac{1}{3},\frac{1}{3},\frac{1}{3})$ with radius $\sqrt{\frac{2}{3}}$)

11) Note ∂S is the unit circle lying on x-y plane.

So we can parametrize ∂S as $(x,y,z) = (\cos\theta, \sin\theta, 0)$ $0 \le \theta \le 2\pi$

So by Stokes' thm,

$$\iint_S (\nabla \times F)\cdot \vec{n}\, dA = \int_{\partial S} \vec{F}\cdot ds$$

$$= \int_0^{2\pi} (\sin\theta, -\cos\theta, 0)\cdot(-\sin\theta, \cos\theta, 0)\, d\theta$$

$$= -2\pi$$

23) a) Note that $F = (x^2, 2xy + x, 0)$ on S.

Hence $\iint_S F\cdot dS = \iint_S (x^2, 2xy+x, 0)\cdot(0,0,1)\, dS = 0$

b) let $c(t) = (\cos t, \sin t, 0)$ be the parametrization of C. Then

$$\int_C F\cdot dS = \int_0^{2\pi} (\cos^2 t, 2\cos t\sin t + \cos t, 0)\cdot(-\sin t, \cos t, 0)\, dt$$

$$= \int_0^{2\pi} (\cos^2 t\sin t + \cos^2 t)\, dt$$

$$= \pi$$

c) $\nabla \times F = (0, 0, 2y+1)$.

$$\iint_S (\nabla\times F)\cdot dS = \int_0^1 \int_0^{2\pi} (0, 0, 2r\sin\theta + 1)\cdot(0,0,1)\, r\, d\theta\, dr$$

$$= \pi$$

Combining results in (b) and (c), Stokes' thm is verified.

25) (i) Direct computation

Parametrize the surface as
$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$z = \frac{1}{2}(x^2+y^2) = \frac{r^2}{2}$$

Since $0 \le z \le 2$, so $0 \le r \le 2$ & $0 \le \theta \le 2\pi$

$T_\theta = (-r\sin\theta, r\cos\theta, 0)$

$T_r = (\cos\theta, \sin\theta, r)$

$T_\theta \times T_r = (r^2\cos\theta, r^2\sin\theta, -r)$ outward normal.

$\nabla \times F = (-\frac{1}{4}r^4 + r\cos\theta, 0, -\frac{1}{2}r^2 - 3)$

$\therefore \iint_S (\nabla \times F) \cdot dS$

$= \int_0^2 \int_0^{2\pi} (\nabla \times F) \cdot (T_\theta \times T_r)\, d\theta\, dr$

$= 20\pi$

Note: need to use $\cos^2\theta = \frac{1+\cos 2\theta}{2}$ in the calculation.

(ii) By Stokes' thm, $\iint_S (\nabla \times F) \cdot dS = \int_{\partial S} F \cdot ds$

Parametrize boundary as $(x, y, z) = (2\cos t, -2\sin t, 2)$ $\quad 0 \le t \le 2\pi$

Use this orientation because the surface lies below the bdry,

so we use clockwise orientation.

$\therefore \int_{\partial S} F \cdot ds = \int_0^{2\pi} (-6\sin t, -4\cos t, 8\sin t) \cdot (-2\sin t, -2\cos t, 0)\, dt$

$= 20\pi$

Note: $\sin^2\theta = \frac{1-\cos 2\theta}{2}$