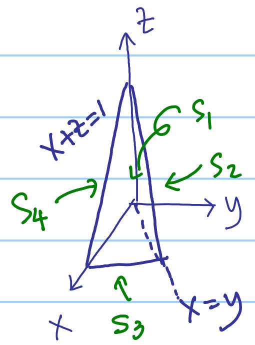
7.5

1)



Since S is union of S_1, S_2, S_3, S_4

So $\iint_S xy\,dS = \iint_{S_1} xy\,dS + \iint_{S_2} xy\,dS + \iint_{S_3} xy\,dS + \iint_{S_4} xy\,dS$

In this question, we will use eqn (5) in P. 478 "Integral over graphs"

let S be the graph of $z = g(x,y)$

$$\iint_S f(x,y,z)\,dS = \iint_D f(x,y,g(x,y)) \cdot \frac{dx\,dy}{\vec{n}\cdot\vec{k}}$$, $\vec{n}$: unit normal of surface

For S_1: we have $z = 1-x \Rightarrow \vec{n} = \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$

$\Rightarrow \vec{n}\cdot\vec{k} = \frac{1}{\sqrt{2}}$

$$\therefore \iint_{S_1} xy\,dS = \int_0^1\int_0^x \sqrt{2}\,xy\,dy\,dx = \frac{\sqrt{2}}{8}$$

For S_2: With domain as $\{0 \le x \le 1, 0 \le z \le 1-x\}$

The graph is $y = g(x,z) = x$

Since $g_x = 1$, $g_z = 0$

$$\iint_{S_2} xy\,dS = \int_0^1\int_0^{1-x} x^2 \cdot \sqrt{2}\,dz\,dx = \frac{\sqrt{2}}{12}$$

For S_3: Since $\vec{n}\cdot\vec{k} = 1$

So $\iint_{S_3} xy\,dS = \int_0^1\int_0^x xy\,dx\,dy = \frac{1}{8}$

For S_4: We have $y=0$ for this surface

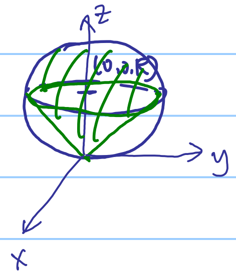
Hence $\iint_{S_4} xy\,dS = 0$

$$\therefore \iint_S xy\,dS = \frac{\sqrt{2}}{8} + \frac{\sqrt{2}}{12} + \frac{1}{8} = \frac{5(\sqrt{2})+3}{24}$$

5) a) Note $x^2+y^2+z^2=2Rz \Leftrightarrow x^2+y^2+(z-R)^2=R^2$

So it is a sphere of radius R, centred at $(0,0,R)$

Parametrize the cone as



$$x=\rho\cos\theta$$
$$y=\rho\sin\theta \qquad 0\le\rho\le R$$
$$z=\rho \qquad 0\le\theta\le 2\pi$$

$$\|T_\rho \times T_\theta\| = \sqrt{2}\rho$$

$$\therefore \text{Area} = \int_0^{2\pi}\int_0^R \sqrt{2}\rho \, d\rho \, d\theta = \sqrt{2}\pi R^2$$

b) Area of the hemisphere of radius R, i.e. $2\pi R^2$

6) Spherical coordinate:
$$x=R\sin\phi\cos\theta$$
$$y=R\sin\phi\sin\theta$$
$$z=R\cos\phi$$

$$T_\phi=(R\cos\phi\cos\theta, R\cos\phi\sin\theta, -R\sin\phi)$$
$$T_\theta=(-R\sin\phi\sin\theta, R\sin\phi\cos\theta, 0)$$

$$T_\phi\times T_\theta=(R^2\sin^2\phi\cos\theta,\ R^2\sin^2\phi\sin\theta,\ R^2\sin\phi\cos\phi)$$

$$\|T_\phi\times T_\theta\| = R^2\sqrt{\sin^4\phi\cos^2\theta+\sin^4\phi\sin^2\theta+\sin^2\phi\cos^2\phi}$$
$$= R^2\sin\phi$$

$$\therefore \|T_\phi\times T_\theta\|\, d\phi\, d\theta = R^2\sin\phi\, d\phi\, d\theta$$

7) Parametrize S : $x = r\cos\theta$ $\quad 0 \le r \le 1$

$y = r\sin\theta$ $\quad 0 \le \theta \le 2\pi$

$z = r^2$

$T_r \times T_\theta = (-2r^2\cos\theta, -2r^2\sin\theta, r)$

$\therefore \|T_r \times T_\theta\| = r\sqrt{4r^2+1}$

$$\iint_S z\,dS = \int_0^{2\pi}\int_0^1 r^2 \cdot r\sqrt{4r^2+1}\,d\theta = 2\pi\int_0^1 r^3\sqrt{4r^2+1}\,dr$$

$$= \frac{5\sqrt{5}\pi}{12} + \frac{\pi}{60}$$

11) a) By replacing x by y, y by z, z by x in $x^2+y^2+z^2=1$

we still get the same equation $x^2+y^2+z^2=1$.

So $\iint_S x^2\,dS = \iint_S y^2\,dS$

Similarly. $\iint_S x^2\,dS = \iint_S z^2\,dS$

b) By (a) $\iint_S (x^2+y^2+z^2)\,dS = 3\iint_S x^2\,dS$

But as $x^2+y^2+z^2 = R^2$, $\iint_S (x^2+y^2+z^2)\,dS = \iint_S R^2\,dS = 4\pi R^4$

so $\iint_S x^2\,dS = \frac{4\pi}{3}R^4$

7.6

1) The heat flux across the surface S is $\iint_S -\nabla T \cdot dS$

Parametrize the surface as $x = \sqrt{2}\cos\theta$, $y = y$, $z = \sqrt{2}\sin\theta$, $\quad 0 \le y \le 2$, $0 \le \theta \le 2\pi$

$$T_\theta \times T_y = (\sqrt{2}\cos\theta, 0, \sqrt{2}\sin\theta)$$

$$\therefore \iint_S -\nabla T \cdot dS = \iint_D -\nabla T \cdot (T_\theta \times T_y)\, dy\, d\theta$$
$$= \int_0^{2\pi}\int_0^2 +6\left((\sqrt{2}\cos\theta)^2 + (\sqrt{2}\sin\theta)^2\right) dy\, d\theta$$
$$= 48\pi$$

3) Write $S = H \cup D$ where H is the upper hemisphere & D is the disk.

$$\iint_S E \cdot dS = \iint_H E \cdot dS + \iint_D E \cdot dS$$

(i) Let $x\vec{i} + y\vec{j} + z\vec{k}$ be the unit normal $\vec{n}$ pointing outward from H.

$$\iint_H E \cdot dS = \iint_H E \cdot n\, dS$$
$$= \iint_H (2x, 2y, 2z) \cdot (x, y, z)\, dS$$
$$= 2\iint_H (x^2 + y^2 + z^2)\, dS$$
$$= 4\pi$$

OR $\quad x = \cos\theta\sin\phi,\ y = \sin\theta\sin\phi,\ z = \cos\phi,\quad 0 \le \theta \le 2\pi,\ 0 \le \phi \le \frac{\pi}{2}$

$$\therefore T_\phi \times T_\theta = (\sin^2\phi\cos\theta,\ \sin^2\phi\sin\theta,\ \sin\phi\cos\phi)$$

$$E \cdot n = 2\cos^2\theta\sin^3\phi + 2\sin^2\theta\sin^3\phi + 2\sin\phi\cos^2\phi$$
$$= 2\sin^3\phi + 2\sin\phi\cos^2\phi = 2(1 - \cos^2\phi)\sin\phi + 2\sin\phi\cos\phi$$
$$= 2\sin\phi$$

$$\iint_H E \cdot dS = 2\int_0^{2\pi}\int_0^{\frac{\pi}{2}} \sin\phi\, d\phi\, d\theta$$
$$= 4\pi$$

(ii) The unit normal is $-\vec{k}$ and $z = 0$ on D.

$$\iint_D E \cdot dS = \iint_D E \cdot n\, dS = \iint_D (2x, 2y, 2z) \cdot (0, 0, -1)\, dS = 0$$

$$\therefore \iint_S E \cdot dS = 4\pi$$

5) The parametrization $\Phi(\phi,\theta)$ for $x^2+y^2+3z^2=1$ is

$$x=\sin\phi\cos\theta$$
$$y=\sin\phi\sin\theta \qquad 0\le\theta\le 2\pi$$
$$z=\frac{1}{\sqrt{3}}\cos\phi \qquad \frac{\pi}{2}\le\phi\le\pi \leftarrow \text{Since } z\le 0$$

$$\nabla\times\vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & -x & zx^3y^2 \end{vmatrix} = (2zx^3y, -3zx^2y^2, -2)$$

$$= \left(\frac{2}{\sqrt{3}}\sin^4\phi\cos^3\theta\sin\theta\cos\phi, \frac{-3}{\sqrt{3}}\sin^4\phi\cos^2\theta\sin^2\theta\cos\phi, -2\right)$$

$$T_\phi = \left(\cos\phi\cos\theta, \cos\phi\sin\theta, -\frac{1}{\sqrt{3}}\sin\phi\right)$$
$$T_\theta = (-\sin\phi\sin\theta, \sin\phi\cos\theta, 0)$$

$$\therefore T_\phi\times T_\theta = \left(\frac{1}{\sqrt{3}}\sin^2\phi\cos\theta, \frac{1}{\sqrt{3}}\sin^2\phi\sin\theta, \sin\phi\cos\phi\right)$$

$$(\nabla\times F)\cdot(T_\phi\times T_\theta) = \frac{2}{3}\sin^6\phi\cos^4\theta\sin\theta\cos\phi - \frac{3}{3}\sin^6\phi\cos^2\theta\sin^3\theta\cos\phi$$
$$-2\sin\phi\cos\phi$$

Note
$$\int_0^{2\pi}\cos^4\theta\sin\theta\,d\theta \qquad \boxed{\begin{array}{c} u=\cos\theta \\ du=-\sin\theta\,d\theta \end{array}}$$
$$=\int_1^1 -u^4\,du$$
$$=0$$

$$\int_0^{2\pi}\cos^2\theta\sin^3\theta\,d\theta$$
$$=\int_0^{2\pi}\cos^2\theta(1-\cos^2\theta)\sin\theta\,d\theta \qquad \boxed{\begin{array}{c} u=\cos\theta \\ du=-\sin\theta\,d\theta \end{array}}$$
$$=\int_1^1 -u^2(1-u^2)\,du$$
$$=0$$

So
$$\iint_S(\nabla\times F)\cdot dS = \int_{\frac{\pi}{2}}^{\pi}\int_0^{2\pi}(-2\sin\phi\cos\phi)\,d\theta\,d\phi \qquad (\because \sin 2\phi = 2\sin\phi\cos\phi)$$
$$=2\pi\int_{\frac{\pi}{2}}^{\pi} -\sin 2\phi\,d\phi$$
$$=\pi\cos 2\phi\Big|_{\frac{\pi}{2}}^{\pi} = 2\pi$$

(1) Parametrize S: $x = \cos\theta \sin\phi$

$y = \sin\theta \sin\phi$ $\quad 0 \le \theta \le 2\pi$

$z = \cos\phi$ $\quad 0 \le \phi \le 2\pi$

$$T_\theta \times T_\phi = (\sin^2\phi \cos\theta, \sin^2\phi \sin\theta, \sin\phi \cos\phi)$$
$$= \sin\phi (\sin\phi \cos\theta, \sin\phi \sin\theta, \cos\phi)$$
$$= \sin\phi (x, y, z)$$

Suppose $\vec{F} = (F_r, F_\theta, F_\phi)$,

then $\vec{F} \cdot \vec{n} = F_r \sin\phi$

$$\therefore \iint_S \vec{F} \cdot dS = \int_0^{2\pi} \int_0^{\pi} F_r \sin\phi \, d\phi \, d\theta$$

The corresponding formula for real-valued function f is

$$\iint_S f \, dS = \int_0^{2\pi} \int_0^{\pi} f \sin\phi \, d\phi \, d\theta$$

(5) Here $\vec{v} \cdot d\vec{S} = \vec{v} \cdot \vec{n} \, dS$ and $\vec{n} = x\vec{i} + y\vec{j} + z\vec{k}$

So $\vec{v} \cdot \vec{n} = x + xy + z^2$

$$\therefore \iint_S v \cdot dS = \iint_S x + xy + z^2 dS$$

By spherical coordinate

$$\iint_S v \cdot dS = \int_0^{\frac{\pi}{2}} \int_0^{2\pi} \left(\sin\phi \cos\theta + (\sin\phi)(\cos\theta)(\sin\theta \sin\phi) + \cos^2\phi\right) \sin\phi \, d\theta \, d\phi$$

$$= \frac{2\pi}{3}$$

17 By the formula for a surface integral over a graph

$$\iint_S \vec{F} \cdot dS = \iint_D \vec{F} \cdot \left(-\frac{\partial g}{\partial x}\vec{i} - \frac{\partial g}{\partial y}\vec{j} + \vec{k}\right) dx\,dy$$

where, in this case, D is the region in the xy-plane determined by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1$$

where $g(x,y) = c\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}$

Thus
$$\iint_S \vec{F} \cdot dS = -\iint_D x^3 \frac{\partial g}{\partial x}\, dx\,dy$$

$$= \frac{c}{a^2}\iint_D \frac{x^4}{\sqrt{1 - (\frac{x}{a})^2 - (\frac{y}{b})^2}}\, dx\,dy$$

$$= \frac{c}{a^2}\int_{-a}^{a}\int_{-b\sqrt{1-\frac{x^2}{a^2}}}^{b\sqrt{1-\frac{x^2}{a^2}}} \frac{x^4}{\sqrt{1 + (\frac{x}{a})^2 + (\frac{y}{b})^2}}\, dy\,dx$$

$$= \frac{2}{5} a^3 b c \pi$$