HW7

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7.1

2) a) $\int_c f(x,y,z)\,ds = \int_c (x+y+z)\,ds$

$$= \int_0^{2\pi} (\sin t + \cos t + 1)\sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}\,dt$$

$$= \int_0^{2\pi} \sqrt{2}\,(\sin t + \cos t + 1)\,dt$$

$$= 2\sqrt{2}\,\pi$$

b) By (a), $ds = \sqrt{2}\,dt$

so $\int_c \cos z\,ds = \int_0^{2\pi} \sqrt{2}\cos t\,dt = 0$

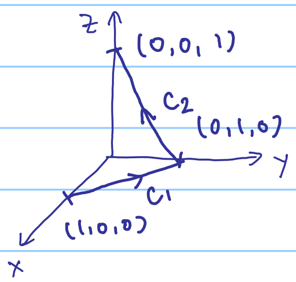
3) a) By definition, $\int_c f\,ds = \int_0^1 e^{\sqrt{t^2}}\sqrt{(2t)^2}\,dt = \int_0^1 2te^t\,dt$

By integration by part, we get

$$\int_0^1 2te^t\,dt = 2\left\{ te^t\Big|_0^1 - \int_0^1 e^t\,dt \right\} = 2(e-(e-1)) = 2$$

7.2

2) c)



Write c as going c_1 first and then c_2

Then we have $\int_c yz\,dx + xz\,dy + xy\,dz$

$$= \int_{c_1} yz\,dx + xz\,dy + xy\,dz + \int_{c_2} yz\,dx + xz\,dy + xy\,dz$$

Parametrization: $c_1(t) = (1-t, t, 0) \quad t\in[0,1]$

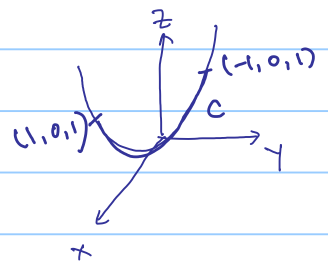
$c_2(t) = (0, 1-t, t) \quad t\in[0,1]$

$\therefore \int_{c_1} yz\,dx + xz\,dy + xy\,dz = \int_0^1 0\,dt = 0$

$\int_{c_2} yz\,dx + xz\,dy + xy\,dz = \int_0^1 0\,dt = 0$

Therefore $\int_c yz\,dx + xz\,dy + xy\,dz = 0$

2, d)



So $c(t) = (t, 0, t^2)$, $-1 \le t \le 1$

$$\int_c x^2 dx - xy\,dy + dz = \int_{-1}^{1} (t^2 dt + 2dt)$$
$$= \int_{-1}^{1} (t^2 + 2t)\, dt$$
$$= \left[\frac{t^3}{3} + t^2\right]\Big|_{-1}^{1}$$
$$= \frac{2}{3},$$

7.3

2) $T_u = (2u)\vec{i} + \vec{j} + (2u)\vec{k}$

$T_v = (-2v)\vec{i} + \vec{j} + 4\vec{k}$

To calculate those vector explicitly at $(-\frac{1}{4}, \frac{1}{2}, 2)$, we must find the point in uv plane corresponding to $(-\frac{1}{4}, \frac{1}{2}, 2)$ on the surface.

To find (u,v), we solve

$$u^2 - v^2 = \frac{1}{4} \quad \cdots (1)$$
$$u + v = \frac{1}{2} \quad \cdots (2)$$
$$u^2 + 4v = 2 \quad \cdots (3)$$

$(3)-(1) \Rightarrow \quad v^2 + 4v = \frac{9}{4}$

i.e. $\quad 4v^2 + 16v - 9 = 0$

$v = \frac{1}{2}$ or $-4\frac{1}{2}$

When $v = \frac{1}{2}$, $u = 0$.

By checking, we know $(0, \frac{1}{2})$ corresponding to $(-\frac{1}{4}, \frac{1}{2}, 2)$

The other possible value of v does not yield a value of u that solves all three equations. A normal to the surface is given by

$$T_u \times T_v = \begin{vmatrix} i & j & k \\ 2u & 1 & 2u \\ -2v & 1 & 4 \end{vmatrix} = (4-2u)\vec{i} + (-8u-4uv)\vec{j} + (2u+2v)\vec{k}$$

At $(u,v) = (0, \frac{1}{2})$, $\quad T_u \times T_v = 4\vec{i} + \vec{k}$

Hence the tangent plane is given by

$$(x-(-\tfrac{1}{4}),\ y-\tfrac{1}{2},\ z-2)\cdot(4,0,1)=0$$

$$4x+z=1$$

15) a) One of the possible choice of parametrization is

$$x = 5\cosh u\cos\theta$$
$$y = 5\cosh u\sin\theta$$
$$z = 5\sinh u$$

b) Since $f(x,y,z)=x^2+y^2-z^2=25$,
the unit normal is $\vec{n}=\frac{\nabla f}{\|\nabla f\|}$, $\nabla f=(2x,2y,2z)$
Thus
$$\vec{n}=\frac{(x,y,-z)}{\sqrt{x^2+y^2+z^2}}$$
$$=\frac{1}{\sqrt{\cosh(2u)}}(\cosh u\cos\theta,\cosh u\sin\theta,-\sinh u)$$

c) Since the normal vector of the tangent plane is parallel to the gradient $\nabla f(x_0,y_0,0)=(2x_0,2y_0,0)$,
the equ of tangent plane is
$$(x_0,y_0,0)\cdot(x-x_0,y-y_0,z-0)=0$$
$$x_0(x-x_0)+y_0(y-y_0)=0$$
$$x_0x+y_0y=25$$

d) First, let's check $(x_0,y_0,0)+t(-y_0,x_0,5)$ lies on the surface.
i.e. we need to check the line satisfies the equation of the surface $x^2+y^2-z^2=25$

$$\text{LHS}=x^2+y^2-z^2=(x_0-ty_0)^2+(y_0+tx_0)^2+(5t)^2$$
$$=x_0^2-2t\cancel{x_0y_0}+t^2y_0^2+y_0^2+2t\cancel{x_0y_0}+t^2x_0^2+25t^2$$
$$=25+t^2(25)-25t^2=25$$

which we have LHS $= 2\pi =$ RHS

Hence this case is done.

The remaining 3 cases can be done by a similar way.

7.4

5) Given $\Phi(u,v) = (u-v,\, u+v,\, uv)$

$$\|T_u \times T_v\| = \sqrt{2}\sqrt{u^2+v^2+2}$$

Since we are integrating over the unit disk, we use polar coordinate.

$$\begin{cases} u = r\cos\theta & 0 \le r \le 1 \\ v = r\sin\theta & 0 \le \theta \le 2\pi \end{cases}$$

Then $\iint_D \sqrt{2}\sqrt{u^2+v^2+2}\, dA = \int_0^1 \int_0^{2\pi} \sqrt{2}\sqrt{r^2+2}\, r\, d\theta\, dr = \frac{\pi}{3}(6\sqrt{6}-8)$

15) $\|T_r \times T_\theta\| = \sqrt{5}\,|\cos\theta|$ ← as we are talking about norm, hence we take the absolute value

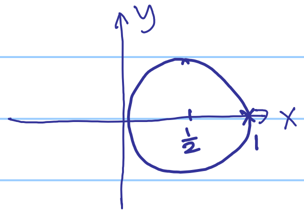
$$A(S) = \int_0^{2\pi}\int_0^1 \sqrt{5}\,|\cos\theta|\, dr\, d\theta$$

$$= \sqrt{5}\int_0^{2\pi} |\cos\theta|\, d\theta$$

$$= \sqrt{5}\left(\int_0^{\frac{\pi}{2}} \cos\theta\, d\theta + \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} (-\cos\theta)\, d\theta + \int_{\frac{3\pi}{2}}^{2\pi} \cos\theta\, d\theta\right)$$

$$= \sqrt{5}(1+2+1) = 4\sqrt{5}$$

17) If we project the cylinder $x^2+y^2=x$ into the xy-plane, it is

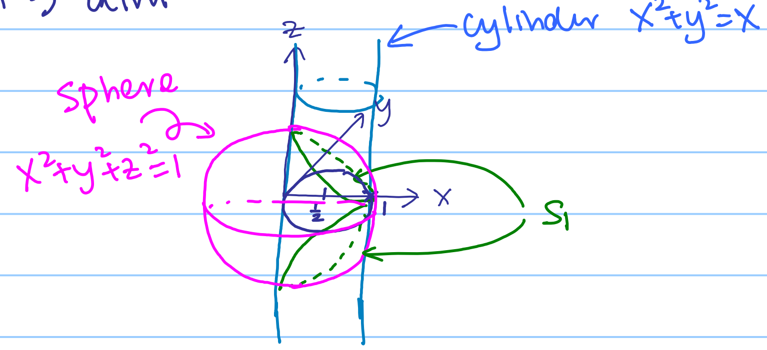


Note: $x^2+y^2=x$

$\Leftrightarrow x^2+y^2-x=0$

$\Leftrightarrow (x-\frac{1}{2})^2+y^2=(\frac{1}{2})^2$ (By completing square)

Hence in 3-dim

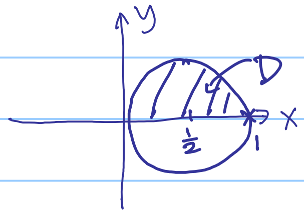


Now we can see S_1 consist of 2 pieces, one in upper hemisphere while the other one in lower hemisphere.

Let's calculate the one in upper hemisphere.

Note equation for sphere in upper hemisphere is

$$z = +\sqrt{1-x^2+y^2} = g(x,y)$$



For the domain, let's us simply consider the shaded area first.

By polar coordinate, $x^2+y^2=x \rightsquigarrow r=\cos\theta$

& $\sqrt{g_x^2+g_y^2+1} = \frac{1}{\sqrt{1-x^2-y^2}} = \frac{1}{\sqrt{1-r^2}}$

By formula in P.465, we have the surface area is given by

$$\iint_D \sqrt{\left(\frac{\partial g}{\partial x}\right)^2+\left(\frac{\partial g}{\partial y}\right)^2+1}\, dA$$

$$= \int_0^{\pi/2}\int_0^{\cos\theta} \frac{r}{\sqrt{1-r^2}}\, dr\, d\theta$$

$$= \frac{\pi}{2} - 1$$

As S_1 consists of 4 pieses of the surface above.

$$A(S_1) = 4\left(\frac{\pi}{2}-1\right) = 2\pi - 4$$

Since surface area of sphere is $4\pi r^2 = 4\pi$

$$\Rightarrow A(S_2) = 4\pi - (2\pi-4) = 2\pi+4$$

$$\therefore \frac{A(S_2)}{A(S_1)} = \frac{2\pi+4}{2\pi-4} = \frac{\pi+2}{\pi-2}$$

(9) b) By formula in P.466,

$$A = 2\pi \int_a^b \left(|x| \sqrt{1+(f'(x))^2} \right) dx \quad \text{with } y = f(x) = mx + q$$

since x is positive between a and b

So $A = 2\pi \int_a^b \left(x\sqrt{1+m^2} \right) dx$

$= \pi (b^2 - a^2)\sqrt{1+m^2}$