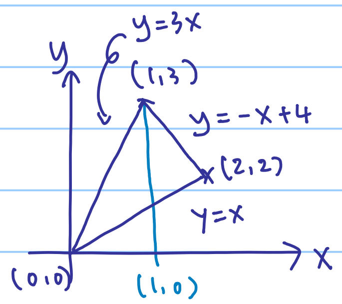
§5.4

10)

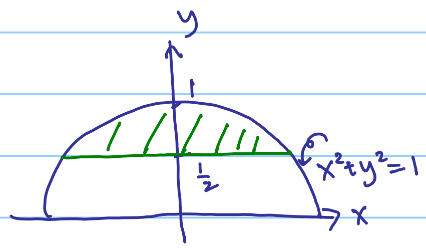


$$\therefore \iint_D e^{x-y}\,dy\,dx = \int_0^1\int_x^{3x} e^{x-y}\,dy\,dx + \int_1^2\int_x^{-x+4} e^{x-y}\,dy\,dx$$

$$= 1+\frac{1}{e^2}$$

Note: in this question, we need to divide the integral into two parts!

11)



$$I = \iint_D y^3(x^2+y^2)^{-3/2}\,dx\,dy$$

$$= \int_{\frac{1}{2}}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} y^3(x^2+y^2)^{-3/2}\,dx\,dy$$

Now, we need to calculate $\int y^3(x^2+y^2)^{-3/2}\,dx$

Sub $x = y\tan\theta$

$dx = y\sec^2\theta\,d\theta$

Then $\displaystyle\int y^3(x^2+y^2)^{-3/2}\,dx$

$$= \int \frac{y^3}{(y^2\tan^2\theta+y^2)^{3/2}}\, y\sec^2\theta\, d\theta$$

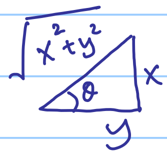
$$= \int \frac{y^4\sec^2\theta}{(y^2\sec^2\theta)^{3/2}}\, d\theta$$

$$= \int \frac{y^4\sec^2\theta}{y^3\sec^3\theta}\, d\theta$$

$$= \int \frac{y}{\sec\theta}\, d\theta$$

$$= \int y\cos\theta\, d\theta$$

$$= y\sin\theta + C$$



Since $x = y\tan\theta$

So $\sin\theta = \dfrac{x}{\sqrt{x^2+y^2}}$

So $\displaystyle\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} y^3(x^2+y^2)^{-3/2}\,dx = \frac{xy}{\sqrt{x^2+y^2}}\Bigg|_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}$

$$= 2y\sqrt{1-y^2}$$

$\therefore$ the original integral I would be

$$\int_{\frac{1}{2}}^{1} 2y\sqrt{1-y^2}\, dy$$

Sub $u = 1-y^2$ when $y=\frac{1}{2}$, $u=\frac{3}{4}$

$du = -2y\,dy$ $y=1$, $u=0$

$$= \int_{3/4}^{0} \sqrt{u}\, du$$

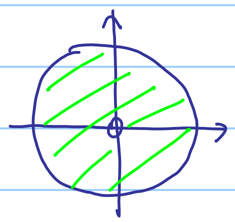
$$= -u^{3/2}\cdot\frac{2}{3}\Big|_{3/4}^{0}$$

$$= \frac{1}{2}$$

DONE ^^

§6.1

1) Image of S = Circle of radius 1, centred at origin, excluding the origin.



if $T(r,\theta)=T(r',\theta')$

$(r\cos\theta, r\sin\theta) = (r'\cos\theta', r'\sin\theta')$

$$\Rightarrow \begin{cases} r\cos\theta = r'\cos\theta' \\ r\sin\theta = r'\sin\theta' \end{cases}$$

So $r^2 = r^2(\cos^2\theta+\sin^2\theta) = (r')^2(\cos^2\theta'+\sin^2\theta') = (r')^2$

$\Rightarrow \quad r^2=(r')^2$

Since r is positive, $r=r'$.

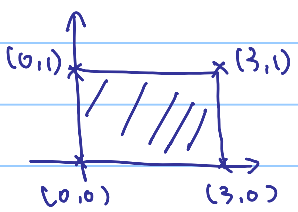
Now as $r\neq 0$, we have $\begin{cases} \sin\theta=\sin\theta' \\ \cos\theta=\cos\theta' \end{cases}$

As θ,θ' lie between 0 and 2π (excluding 2π)

$\Rightarrow \quad \theta=\theta'$

3) Hint for figuring out the image: Check vertex to get a sense of it

$\therefore$ Image of T



T is 1-1 because if $T(u,v)=T(u',v')$

$(-u^2+4u, v) = (-(u')^2+4u', v')$

$\Rightarrow v=v'$ & $-u^2+4u = -(u')^2+4u'$

$\Rightarrow (u')^2-u^2 = 4(u'-u)$

$\Rightarrow (u'-u)(u'+u) = 4(u'-u)$

$\Rightarrow \quad (u'+u-4)(u'-u)=0$

Then we have $\quad u'+u-4=0 \quad$ or $\quad u'-u=0$

But it is impossible for $u'+u-4=0$ as $u', u \leq 1$

$\Rightarrow \quad u'=u$

Now we have $\quad u=u'$ & $v=v'$

So T is 1-1.

§6.2

5) $T: D^* \longrightarrow D, \quad T(u,v)=(u, \frac{v}{2})$

$D=[0,1]\times[0,1]$

$\Rightarrow D^*=[0,1]\times[0,2]$

and $\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \begin{vmatrix} 1 & 0 \\ 0 & 1/2 \end{vmatrix} = \frac{1}{2}$

So $\displaystyle \iint_D \frac{dx\,dy}{\sqrt{1+x+2y}} = \int_0^1\int_0^2 \frac{1}{\sqrt{1+u+v}}\cdot\frac{1}{2}\,dv\,du$

$= \frac{2}{3}(9-2\sqrt{2}-3\sqrt{3})$

9) Use polar coordinates,

so $\displaystyle \iint_D (x^2+y^2)^{3/2}\,dx\,dy = \int_0^2\int_0^{2\pi} r^3\cdot r\,d\theta\,dr = \frac{64\pi}{5}$

11) $\displaystyle \text{Area} = \int_0^{2\pi}\int_0^{1+\sin\theta} r\,dr\,d\theta$

$\displaystyle = \int_0^{2\pi}\frac{(1+\sin\theta)^2}{2}\,d\theta$

$\displaystyle = \int_0^{2\pi}\left(\frac{1}{2}+\sin\theta+\underbrace{\frac{\sin^2\theta}{2}}\right)d\theta$ use double angle formula

$\displaystyle = \int_0^{2\pi}\left(\frac{1}{2}+\sin\theta+\boxed{\frac{1-\cos 2\theta}{4}}\right)d\theta$

$= \frac{3\pi}{2}$

13) Use Cylinderical Coord.

Then we have $\int_2^3 \int_0^2 \int_0^{2\pi} z e^{r^2} \cdot r \, d\theta \, dr \, dz$

$$= \int_2^3 \int_0^2 2\pi z e^{r^2} r \, dr \, dz$$

Use $u = r^2$, $du = 2r\,dr$

$$= \int_2^3 \int_0^4 \pi z e^u \, du \, dz$$

$$= \int_2^3 \pi z e^u \Big|_0^4 \, dz$$

$$= \frac{5\pi}{2}(e^4 - 1)$$

23) By use of spherical coord.

$$\iiint_W \frac{dx\,dy\,dz}{(x^2+y^2+z^2)^{3/2}}$$

$$= \int_0^\pi \int_0^{2\pi} \int_a^b \frac{\rho^2 \sin\phi}{\rho^3} \, d\rho \, d\theta \, d\phi$$

$$= 4\pi \log\left(\frac{b}{a}\right)$$