8.3

2) a) Since $y=2x^2$. Parametrize c as $(x,y)=(x,2x^2)$, $0\le x\le 1$

$\int_c F\cdot ds = \int_0^1 (2x^3+16x^5)\,dx = \frac{19}{6}$

b) Yes, since $\nabla\times F = (0,0,-1) \neq 0$

3) If $F=\nabla f$, then

$\frac{\partial f}{\partial x} = 2xyz+\sin x$ (1)

$\frac{\partial f}{\partial y} = x^2 z$ (2)

$\frac{\partial f}{\partial z} = x^2 y$ (3)

Integrate (1) with respect to x

$f = x^2yz - \cos x + h(y,z)$ where h is a fcn of y & z

Integrate (2) with respect to y

$f = x^2yz + g(x,z)$ where g is a fcn of x & z

Integrate (3) with respect to z

$f = x^2yz + k(x,y)$ where k is a fcn of x & y.

$\therefore$ $g(x,z) = k(x,y) = -\cos x + C$ where C is a constant

$h(y,z) = C$

$\therefore$ $f(x,y,z) = x^2yz - \cos x + C$

6) a) $\frac{\partial}{\partial x}\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right) = \frac{-x}{(x^2+y^2+z^2)^{3/2}}$

Repeat for $\frac{\partial}{\partial y}$, $\frac{\partial}{\partial z}$

$\nabla\left(\frac{1}{r}\right) = -\frac{\vec{r}}{r^3}$

b) Since $\vec{F}$ is the gradient of a function f

$\int_c \vec{F}\cdot ds = f(c(a)) - f(c(b))$ for all paths $\vec{c}(t)$, $a\le t\le b$

unless $\vec{c}$ passes through the origin, where $\frac{1}{r}$ is not continuous

$\therefore$ work done by $\vec{F}$ is moving a particle from r_0 "to ∞" is

$\frac{1}{\sqrt{x^2+y^2+z^2}} - \lim_{r\to\infty}\frac{1}{r} = \frac{1}{\sqrt{x^2+y^2+z^2}}$

9) Note $\nabla \times \vec{F} = 0$

By integrating as in question 3, $f(x,y,z) = e^x \sin y + \frac{z^3}{3}$

and $c(0) = (0,0,1)$, $c(1) = (1,1,e)$.

$\therefore \int_C \vec{F} \cdot ds = f(c(1)) - f(c(0)) = e \sin 1 + \frac{e^3}{3} - \frac{1}{3}$

(2) a) Let $\vec{c}(x,y) = (\cos\theta, \sin\theta)$ $\quad 0 \le \theta < 2\pi$

$\therefore \int_C \frac{x\,dy - y\,dx}{x^2+y^2} = \int_0^{2\pi} \frac{1}{1}\left(\cos\theta(\cos\theta - \sin\theta(-\sin\theta)\right)d\theta$

$= \int_0^{2\pi} 1\, d\theta = 2\pi$

b) Since C is a closed curve, but $\int_C F \cdot ds \neq 0$

So F is not a conservative vector field.

c) No, it is because $\vec{F}$ is not continuous at the origin that it does not satisfies the condition in Thm 7.

(4) a) Yes. Since $\frac{\partial F_1}{\partial y} - \frac{\partial F_2}{\partial x} = x\sin xy - x\sin xy - x^2 y\cos xy$

$- (-2x\sin xy - x^2 y \cos xy)$

$= 0$

One of the choice of f s.t. $F = \nabla f$: $f(x,y) = x\cos xy$

b) No $\quad$ By $\frac{\partial F_1}{\partial y} - \frac{\partial F_2}{\partial x} \neq 0$ (need to compute yourselves)

c) Yes $\quad$ Again because of $\frac{\partial F_1}{\partial y} - \frac{\partial F_2}{\partial x} = 0$ by computation

$f(x,y) = x^2\cos y + x\cos y$ is one of the f s.t. $\nabla f = F$

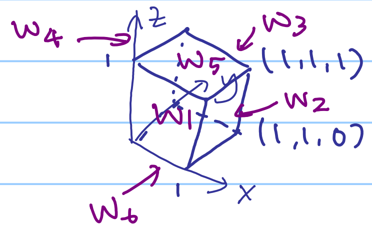
8.4

1) $\operatorname{div} F = 1+1+1 = 3$

$\therefore$ By divergence thm

$$\iint_{\text{unit sphere}} F \cdot dS$$
$$= \iiint_{\text{unit ball}} (\nabla \cdot F)\, dV$$
$$= \int_0^1 \int_0^{2\pi} \int_0^{\pi} 3\, \rho^2 \sin\phi \, d\phi\, d\theta\, d\rho$$
$$= 2\pi(2) = 4\pi$$

3) W :



(i)
$$\iint_{W_1} F \cdot dS = \int_0^1 \int_0^1 (x,0,z)\cdot(0,-1,0)\,dx\,dz = 0$$
$$\iint_{W_2} F \cdot dS = \int_0^1 \int_0^1 (1,y,z)\cdot(1,0,0)\,dy\,dz = \int_0^1\int_0^1 1\,dy\,dz = 1$$
$$\iint_{W_3} F \cdot dS = \int_0^1 \int_0^1 (x,1,z)\cdot(0,1,0)\,dx\,dz = 1$$
$$\iint_{W_4} F \cdot dS = \int_0^1 \int_0^1 (0,y,z)\cdot(-1,0,0)\,dy\,dz = 0$$
$$\iint_{W_5} F \cdot dS = \int_0^1 \int_0^1 (x,y,1)\cdot(0,0,1)\,dx\,dy = 1$$
$$\iint_{W_6} F \cdot dS = \int_0^1 \int_0^1 (x,y,0)\cdot(0,0,-1)\,dx\,dy = 0$$
$$\therefore \iint_{\partial W} F \cdot dS = 3$$

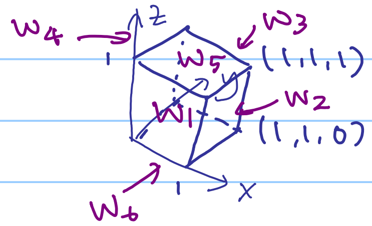
(ii) By divergence thm,
$$\iint_{\partial W} F \cdot dS = \iiint_W (\nabla \cdot F)\,dV = \iiint_W (1+1+1)\,dV = 3 \cdot (\text{vol of unit cube})$$
$$= 3$$

7) Note $\nabla \cdot F = 1+1+0 = 2$

By divergence thm, flux is
$$\iiint_W (\nabla\cdot F)\,dV = \iiint_W 2\,dV$$
$$= 2\cdot(\text{vol of } W)$$
$$= 2\cdot(1\cdot 1\cdot 3) = 6$$

9) (i) Directly:

W :



$$\vec{F} = (x, y, -z)$$

$$\iint_{W_1} F \cdot dS = \int_0^1 \int_0^1 (x, 0, -z) \cdot (0, -1, 0)\, dx\, dz = 0$$

$$\iint_{W_2} F \cdot dS = \int_0^1 \int_0^1 (1, y, -z) \cdot (1, 0, 0)\, dy\, dz = \int_0^1 \int_0^1 1\, dy\, dz = 1$$

$$\iint_{W_3} F \cdot dS = \int_0^1 \int_0^1 (x, 1, -z) \cdot (0, 1, 0)\, dx\, dz = 1$$

$$\iint_{W_4} F \cdot dS = \int_0^1 \int_0^1 (0, y, -z) \cdot (-1, 0, 0)\, dy\, dz = 0$$

$$\iint_{W_5} F \cdot dS = \int_0^1 \int_0^1 (x, y, -1) \cdot (0, 0, 1)\, dx\, dy = -1$$

$$\iint_{W_6} F \cdot dS = \int_0^1 \int_0^1 (x, y, 0) \cdot (0, 0, -1)\, dx\, dy = 0$$

$$\therefore \iint_{\partial W} F \cdot dS = 1$$

(ii) Note $\nabla \cdot F = 1 + 1 - 1 = 1$

By div thm,

$$\iint_{\partial W} F \cdot dS = \iiint_W (\nabla \cdot F)\, dV = \iiint_{\partial W} 1\, dV = 1$$

(3) Since $\nabla \cdot \left(\frac{\vec{r}}{r^2}\right) = \frac{1}{r^2}$ By computation

If $(0,0,0) \notin \Omega$, by div. thm, $\iiint_\Omega \frac{1}{r^2} dV = \iint_{\partial W} (\vec{r} \cdot \vec{n}) \cdot \frac{1}{r^2} dS$

If $(0,0,0) \in \partial\Omega$, let $B_\varepsilon = \{(x, y, z) \mid \sqrt{x^2+y^2+z^2} < \varepsilon\}$

$$\begin{aligned}
\iiint_\Omega \frac{1}{r^2} dV &= \lim_{\varepsilon \to 0} \iiint_{\Omega - B_\varepsilon} \frac{1}{r^2} dV \\
&= \lim_{\varepsilon \to 0} \iint_{\partial(\Omega \setminus B_\varepsilon)} \frac{\vec{r} \cdot \vec{n}}{r^2} dS \\
&= \lim_{\varepsilon \to 0} \left(\iint_{\partial\Omega} \frac{\vec{r} \cdot \vec{n}}{r^2} dS - \iint_{\partial B_\varepsilon} \frac{\vec{r} \cdot \vec{n}}{r^2} dS \right) \\
&= \lim_{\varepsilon \to 0} \left(\iint_{\partial\Omega} \frac{\vec{r} \cdot \vec{n}}{r^2} dS - 4\pi\varepsilon \right) \\
&= \iint_{\partial\Omega} \frac{\vec{r} \cdot \vec{n}}{r^2} dS
\end{aligned}$$

23) By div. thm

$\iint_S \vec{r}\cdot\vec{n}\,dS = \iiint_W \nabla\cdot\vec{r}\,dV = 3\iiint_W dV = 3(\text{Volume of } W)$

Geometric interpretation:

Assume $(0,0,0)\in W$ and consider the skew cone with its vertex at $(0,0,0)$ with base ΔS and altitude $\|r\|$.

Its volume is $\frac{1}{3}(\Delta S)(\vec{r}\cdot\vec{n})$

By Riemann sum, Vol of $W = \frac{1}{3}\iint_S \vec{r}\cdot\vec{n}\,dS$