Ch.8 The Integral Theorem of Vector Analysis

8.1 Green's Theorem
1. **Green's Theorem**

For a simple region D with bounding curve $C = \partial D$ and two C^1 function P and Q on D, we have

$$\int_C P dx + Q dy = \int\int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx dy$$

2. **Area** Area of a region is

$$A = \frac{1}{2}\int_{\partial D} x dy - y dx.$$

8.2 Stokes' Theorem
1. **Stokes' Theorem**

Let S be the oriented surface defined by the graph of a C^2 function $z = f(x, y)$, where $(x, y) \in D$, a region in the plane to which Green's theorem applies, and let F be a C^1 vector field on a region containing the surface. If ∂S denotes the oriented boundary curve of S, then

$$\int\int_S curl\ \mathbf{F}\ \cdot d\mathbf{S} = \int_{\partial S} \mathbf{F} \cdot d\mathbf{s}$$

8.3 Conservative Fields
1. Let **F** be a C^1 vector field defined on $\mathbb{R}^3$ except possibly for a finite number of points. The following conditions on **F** are equivalent:

  a. For any oriented simple closed curve C, $\int_C \mathbf{F} \cdot d\mathbf{s} = 0$
  b. For any two oriented simple curves C_1, C_2 that have the same endpoints,
$$\int_{C_1} \mathbf{F} \cdot d\mathbf{s} = \int_{C_2} \mathbf{F} \cdot d\mathbf{s}$$
  c. **F** is the gradient of some function f ; that is $F = \nabla f$
  d. $\nabla \times \mathbf{F} = 0$

2. Planar case. In the plane, a vector field $\mathbf{F} = \mathbf{P}i + \mathbf{Q}j$ defined and C^1 everywhere, is a gradient if and only if

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

8.4 Gauss' Theorem

1. **Gauss' Divergence Theorem** For a region W with boundary ∂W oriented by the outward pointing unit normal and if **F** is a smooth vector field defined on W, then

$$\iiint_W (\nabla \cdot \boldsymbol{F})dV = \iint_{\partial W} \boldsymbol{F} \cdot d\boldsymbol{S}$$