Ch. 7 Integrals over Paths and Surfaces

7.1 The Path Integral

1. The **path integral** of scalar function f(x,y,z) along a path **c**(t), where $a \leq t \leq b$ is defined by

$$\int_{c} f\ ds = \int_{a}^{b} f\big(x(t), y(t), z(t)\big)||\boldsymbol{c}'(t)||\, dt.$$

2. **Arc length:** If $f = 1$, then the definition of the path integral reduces to that for the arc length of the path.

7.2 Line Integrals

The **line integral** of a given continuous vector field **F** along a path **c**(t), where $a \leq t \leq b$, is defined by

$$\int_{c} F \cdot ds = \int_{a}^{b} F\big(c(t)\big) \cdot c'(t)\, dt.$$

7.3 Parameterized Surfaces

1. A **parametrized surface** is a map

$$\phi\colon D \to \mathbb{R}^3$$

Written as

$$\phi(u, v) = \big(x(u, v), y(u, v), z(u, v)\big).$$

2. Tangent vectors to the surface are given by

$$T_u = \frac{\partial \phi}{\partial u} = \frac{\partial x}{\partial u}\boldsymbol{i} + \frac{\partial y}{\partial u}\boldsymbol{j} + \frac{\partial z}{\partial u}\boldsymbol{k}$$

$$T_v = \frac{\partial \phi}{\partial v} = \frac{\partial x}{\partial v}\boldsymbol{i} + \frac{\partial y}{\partial v}\boldsymbol{j} + \frac{\partial z}{\partial v}\boldsymbol{k}$$

With a normal vector given by

$$\mathbf{n} = T_u \times T_v$$

3. A surface is called **regular** if $T_u \times T_v \neq 0$. Then the tangent plane at a point (x_0, y_0, z_0) on the surface is given by

$$(x - x_0, y - y_0, z - z_0) \cdot \boldsymbol{n} = 0,$$

where the normal vector **n** is evaluated at the point $\phi(u_0, v_0) = (x_0, y_0, z_0)$.

4. Important parametrization.

a. Circle of radius r

$$x = r\cos\theta\,,\quad y = r\sin\theta\,,\quad 0 \le \theta \le 2\,\pi$$

b. Hyperbola $x^2 + y^2 = 1$

$$x = \cosh t\,,\qquad y = \sinh t$$

c. Sphere of radius ρ

$$x = \rho\cos\theta\sin\phi\,,\qquad y = \rho\sin\theta\sin\phi\,,\qquad z = \rho\cos\phi\,,$$
$$0 \le \theta < 2\,\pi, 0 \le \phi < \pi$$

d. A surface $z = g(x, y)$

$$x = u,\qquad y = v,\qquad z = g(u, v)$$

5. Normal to a graph

If $z = f(x, y)$, the normal to the graph is $\left(-\frac{\partial f}{\partial x}, -\frac{\partial f}{\partial y}, 1\right)$. Note that if we let $g = z - f(x, y)$, then the normal to the surface is $\mathbf{n} = \nabla g$.

7.4 Area of a Surface

1. **Area of a parametrized surface**:

$$A(S) = \int\int_D \|T_u \times T_v\| dudv$$
$$= \int\int_D \sqrt{\left[\frac{\partial(y,z)}{\partial(u,v)}\right]^2 + \left[\frac{\partial(x,y)}{\partial(u,v)}\right]^2 + \left[\frac{\partial(x,z)}{\partial(u,v)}\right]^2}$$

2. Sphere. $x^2 + y^2 + z^2 = R^2$, the scalar surface element is given by:

$$dS = R^2 \sin\phi\;\, d\phi d\theta$$

3. Graph. $z = g(x, y)$ can be parametrized by

$$x = u,\qquad y = v,\qquad z = g(u, v)$$

4. **Surface area of a graph**

$$A(S) = \int\int_D \left(\sqrt{\left[\frac{\partial g}{\partial x}\right]^2 + \left[\frac{\partial g}{\partial x}\right]^2 + 1}\right) dudv$$

5. **Surfaces of Revolution**

a. Revolve $y = f(x)$, where $a \le x \le b$, about the x-axis:

$$A(S) = 2\pi \int_a^b \left(|f(x)| \sqrt{1 + \left(f'(x)\right)^2} \right) dx$$

b. Revolve $y = f(x)$, where $a \leq x \leq b$, about the y-axis:

$$A(S) = 2\pi \int_a^b \left(|x| \sqrt{1 + \left(f'(x)\right)^2} \right) dx$$

7.5 Integrals of Scalar Functions over Surfaces

1. **Scalar Surface Integral**

$$\int\int_S f\,dS = \int\int_D f\left(x(u,v), y(u,v), z(u,v)\right) \left\| T_u \times T_v \right\| du\,dv$$

2. **Graph**. For $z = g(x, y)$ with $\phi(u, v) = \left(u, v, g(u, v)\right)$,

$$T_u = \boldsymbol{i} + \frac{\partial g}{\partial u}\boldsymbol{k}\,;\;\; T_v = \boldsymbol{j} + \frac{\partial g}{\partial v}\boldsymbol{k}$$

$$T_u \times T_v = \begin{vmatrix} i & j & k \\ 1 & 0 & \dfrac{\partial g}{\partial u} \\ 0 & 1 & \dfrac{\partial g}{\partial v} \end{vmatrix} = -\frac{\partial g}{\partial u}\boldsymbol{i} - \frac{\partial g}{\partial v}\boldsymbol{j} + \boldsymbol{k}$$

3. **Scalar Surface Element Formulas**

a. Parametrized Surface

$$dS = \left\| T_u \times T_v \right\| du\,dv$$

b. Graph

$$dS = \left(\sqrt{\left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2 + 1} \right) dx\,dy$$

c. Plane. If S is a plane,

$$dS = \frac{dxdy}{\cos\theta} = \frac{dxdy}{\boldsymbol{n} \cdot \boldsymbol{k}}$$

where $\cos\theta = \boldsymbol{n} \cdot \boldsymbol{k}$, and **n** is the upward pointing unit normal vector to the surface.

d. Sphere $x^2 + y^2 + z^2 = R^2$:

$$dS = R^2 \sin\phi\, d\phi\, d\theta$$

7.6 Surface Integrals of Vector Functions

1. The formula for the surface integral of a vector field **F** over a parametrized surface is given by:

$$\int\int_S \boldsymbol{F} \cdot d\boldsymbol{S} = \int\int_D F \cdot (T_u \times T_v) dudv$$

2. Vector Surface Element for a Sphere of Radius R:

$$\mathrm{d}\mathbf{S} = (\mathrm{x}\mathbf{i} + \mathrm{y}\mathbf{j} + \mathrm{z}\mathbf{k})\mathrm{R}\sin\phi\, \mathrm{d}\phi\mathrm{d}\theta$$

3. Graphs. If S is a graph, $\mathrm{z} = \mathrm{g}(\mathrm{x}, \mathrm{y})$, the default orientation is the upward normal.

$$\mathrm{dS} = \left(-\frac{\partial \mathrm{g}}{\partial \mathrm{x}}\boldsymbol{i} - \frac{\partial g}{\partial y}\boldsymbol{j} + \boldsymbol{k}\right) dxdy$$

$$\int\int_S \boldsymbol{F} \cdot d\boldsymbol{S} = \int\int_D \left[F_1\left(-\frac{\partial g}{\partial x}\right) + F_2\left(-\frac{\partial g}{\partial x}\right) + F_3\right] dxdy.$$

Good Luck,

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