Ch. 6 The Change of Variables Formula and Applications

6.1 The Geometry of Maps from $\mathbb{R}^2$ to $\mathbb{R}^2$

1. A **mapping** T of a region D* in $\mathbb{R}^2$ to $\mathbb{R}^2$ associates to each point $(\mathrm{u},\mathrm{v}) \in \mathrm{D}^*$ a point $(\mathrm{x},\mathrm{y}) = \mathrm{T}(\mathrm{u},\mathrm{v})$. The set of all such (x,y) is the **image** domain $\mathrm{D} = \mathrm{T}(\mathrm{D}^*)$.
2. If T is **linear**, i.e. if $\mathrm{T}(\mathrm{u},\mathrm{v}) = A\begin{bmatrix} u \\ v \end{bmatrix}$, where A is a 2×2matrix, then T maps parallelograms to parallelograms, mapping the sides and vertices in the domain to those in the image.
3. A map T is called **one-to-one** if different points get sent to different points.
4. If T is linear, determined by a 2×2matrix A, then T is one-to-one when $\det A \neq 0$.
5. When D is the image of T; that is, $\mathrm{D} = \mathrm{T}(\mathrm{D}^*)$, we say T maps D^* **onto** D.

6.2 The Change of Variables Theorem

1. The **Jacobian determinant** of a C^1 mapping $\mathrm{T}: \mathrm{D}^* \subset \mathbb{R}^2 \to \mathbb{R}$;
$T(u,v) = \big(x(u,v), y(u,v)\big)$ is defined by
$$\frac{\partial(\mathrm{x},\mathrm{y})}{\partial(\mathrm{u},\mathrm{v})} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$
2. The two-variable change of variables formula states for a C^1 map $\tau: \mathrm{D}^* \to D$ that is one-to-one and onto D, and an integrable function $\mathrm{f}: \mathrm{D} \to \mathbb{R}$,
$$\int\int_D f(x,y)dxdy = \int\int_{D^*} f\big(x(u,v), y(u,v)\big)\left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv.$$
3. Polar Coordinates: $\mathrm{x} = \mathrm{r}\cos\theta,\ \mathrm{y} = \mathrm{r}\sin\theta$, the change of variable formula reads
$$\int\int_D f(x,y)dxdy = \int\int_{D^*} f(r\cos\theta, r\sin\theta)\, r\, drd\theta$$
4. The triple-variable change of variables formula states for a C^1 map $\tau: \mathrm{W}^* \to W$ that is one-to-one and onto W, and an integrable function $\mathrm{f}: \mathrm{W} \to \mathbb{R}$,
$$\int\int\int_W f(x,y,z)dxdydz$$
$$= \int\int\int_{W^*} f(x(u,v,w), y(u,v,w), z(u,v,w))\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| dudvdw.$$
5. Cylindrical Coordinates. For $\mathrm{x} = \mathrm{r}\cos\theta,\ \mathrm{y} = \mathrm{r}\sin\theta, z = z$,
$$\int\int\int_W f(x,y,z)dxdydz = \int\int\int_{W^*} f(r\cos\theta, r\sin\theta, z)\ r\, drd\theta dz$$
6. Spherical Coordinates. For $\mathrm{x} = \rho\sin\phi\cos\theta,\ y = \rho\sin\phi\sin\theta, z = \rho\cos\phi$,
$$\int\int\int_W f(x,y,z)dxdydz$$
$$= \int\int\int_{W^*} f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\, \rho^2\sin\phi\, d\rho\, d\theta d\,\phi$$