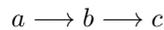


MATH 115 Problem Set 7

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1. Let $N_a(t)$, $N_b(t)$, $N_c(t)$ representation the numbers of nuclei of three radioactive substances which may decay according to the scheme

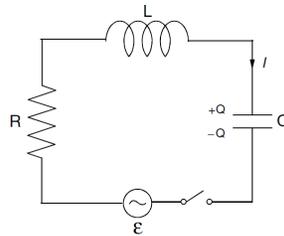


with decay constants λ_a and λ_b where the substance c is considered stable. Then the functions are known to obey the system of differential equations

$$\begin{aligned}\frac{dN_a}{dt} &= -\lambda_a N_a, \\ \frac{dN_b}{dt} &= -\lambda_a N_b + \lambda_a N_a, \\ \frac{dN_c}{dt} &= -\lambda_b N_b,\end{aligned}$$

Assuming $N_a(0) = N_0$ and $N_b(0) = N_c(0) = 0$, find $N_a(t)$, $N_b(t)$ and $N_c(t)$ as functions in time t .

2. **Simple electric circuit** Consider the simple electric circuit shown below



It consists of three kinds of circuit elements; a resistor with a resistance R measured in ohms, an inductor with an inductance L measured in henries, and a capacitor with capacitance C measured in farads. They are connected in series with a source of electromotive force (emf) that supplies at time t a voltage $V(t)$ measured in volts. The capacitor is a device to store electric charges Q , measured in coulombs. If the switch is closed, there will be current $I(t)$, measured in amperes, flowing in the circuit. In elementary physics, we learned that the voltage drop across the resistor is equal to IR , the voltage drop across the inductor is $L \frac{dI}{dt}$, and the voltage across the capacitor is $\frac{1}{C}Q$. The sum of these is equal to the applied voltage. Therefore

$$L \frac{dI}{dt} + RI + \frac{1}{C}Q = V(t).$$

Furthermore, the rate of increase of the charge Q on the capacitor is, by definition, equal to the current $\frac{dQ}{dt} = I$. With this relation, we obtain the following second-order linear nonhomogeneous equation for Q :

$$L \frac{d^2Q}{dt^2} + R \frac{dQ}{dt} + \frac{1}{C}Q = V(t).$$

Suppose the circuit is driven by a generator with a pure cosine wave oscillation, $V(t) = V_0 \cos \omega t$, then the equation becomes

$$L \frac{d^2 Q}{dt^2} + R \frac{dQ}{dt} + \frac{1}{C} Q = V_0 \cos \omega t.$$

Solve the above differential equation to obtain $Q(t)$ and $I(t)$.

3. Find the Laplace transformation of each of the following functions by using the 'Multiply t' operation

(a) te^t ,

(b) $t \cos t$.

4. Find the Laplace transformation of

$$\frac{1}{t}(e^{2t} - e^{-2t})$$

by using the 'Divide t' operation.

5. Find the Laplace transformation of

$$\sinh t \cos t$$

by using the 'Shifting -s' operation.

6. Show that

$$\mathcal{L}[-t \cos at] = \frac{1}{s^2 + a^2} - \frac{2s^2}{(s^2 + a^2)^2}.$$