MATH 115 Problem Set 10

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1. Show that if L is a linear operator and

$$\langle h, Lh\rangle = \langle Lh, h\rangle$$

for all function h in the complex function space, then L is Hermitian.

2. (a) Show that the following differential equation together with the boundary conditions is a Sturm-Liouville problem:

$$y'' - 2y' + \lambda y = 0, \qquad 0 \le x \le 1, \quad y(0) = 0, y(1) = 0.$$

What is the weight function w?

(b) Determine the eigenvalues and corresponding eigen-functions of the problem.

3. (a) Show that the Chebyshev equation

$$(1 - x^2)y''(x) - xy'(x) + \lambda y(x) = 0, \quad -1 \le x \le 1$$

can be converted into

$$\frac{d^2}{d\theta^2}\Theta(\theta) + \lambda\Theta(\theta) = 0, \qquad 0 \le \theta \le \pi$$

with a change of variable $x = \cos\theta$, where $\Theta(\theta) = y(\cos\theta)$.

(b) Show that in terms of θ, $\frac{dy}{dx}$ can be expressed as

$$\frac{dy}{dx} = \left(A\sqrt{\lambda}\sin\sqrt{\lambda}\theta - B\sqrt{\lambda}\cos\sqrt{\lambda}\right)\frac{1}{\sin\theta}$$

(c) Show that the condition for y and $y'(x)$ to be bounded are

$$B = 0, \qquad \lambda = n^2, n = 0, 1, 2, \ldots$$

Therefore the engenfunctions are $\Theta(theta) = \cos n\theta$ with eigenvalue $\lambda_n = n^2$.

(d) Define $T_n(x)$ by $T_n(\cos\theta) = \Theta_n(\theta) = \cos n\theta$. Show that for any integers n and m

$$\int_{-1}^{1} T_n(x)T_m(x)\frac{1}{\sqrt{1-x^2}}dx = \begin{cases} 0 & n \ne m \\ \pi & n = m = 0 \\ \frac{\pi}{2} & n = m \ne 0. \end{cases}$$

4. Find the eigenfuncton expansion for the solution with boundary conditions $y(0) = y(\pi) = 0$ of the inhomogeneous differential

$$y'' + ky = f(x)$$

where k is a constant and

$$f(x) = \begin{cases} x & 0 \le x \le \pi/2 \\ \pi - x & \pi/2 \le x \le \pi. \end{cases}$$

5. Solve the differential equation

$$\frac{df(t)}{dt} + \alpha f(t) = 0$$

by expanding $f(t)$ into a Frobenius series $f(t) = t^p \sum_{n=0}^{\infty} a_n t^n$.

6. (a) Show that an equation of the form

$$x^2 y''(x) + xy'(x) + (\alpha x^\beta)^2 y - b^2 y = 0$$

is transformed into a Bessel equation

$$x^2 y''(z) + zy'(z) + z^2 y - \left(\frac{b}{\beta}\right)^2 y = 0$$

by a change of variable $z = \frac{ax^\beta}{\beta}$.

(b) Solve

$$x^2 y'' + xy' + 4x^4 y - 16y = 0.$$

7. Laguerre Polynomials

(a) Use the Frobenius method to solve the Laguerre equation

$$xy'' + (1 - x)y' + \lambda y = 0.$$

(b) Show that if λ is a nonnegative integer n, then the solution is a polynomial of order n. If the polynomial is normalized such that it is equal to 1 at $x = 0$, it is known as the Lauerre polynomial $L_n(x)$. Show that

$$L_n(x) = \sum_{k=0}^{n} \frac{(-1)^k n!}{(n-k)!(k!)^2} x^k.$$