Math 10a Week 10

§ 4.1 Using first & second derivative

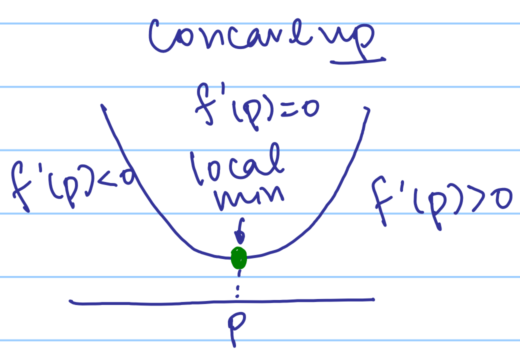
1. Suppose p is a point in the domain of f:
   - f has a local min at p if $f(p) \le$ values of f for pt. near p.
   - f has a local max at p if $f(p) \ge$ values of f for pt. near p.

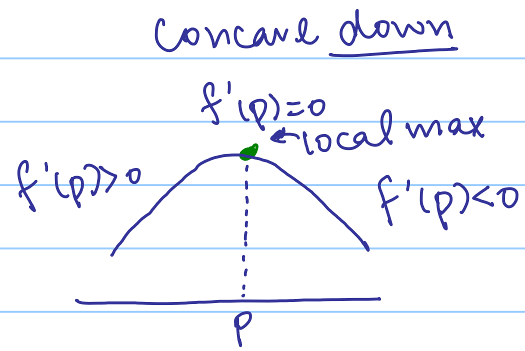
2. Critical pt. is a point p in the domain such that $f'(p)=0$
   critical value $= f(p)$, where $f'(p)=0$

3. First-Derivative Test for Local Maxima & Minima.
   Suppose p is a critical pt. of a continuous at f.
   - If f' changes from ⊖ to ⊕ at p, then f has a local min at p.
   - If f' changes from ⊕ to ⊖ at p, then f has a local max at p.

4. Second Derivative Test.
   - If $f'(p)=0$ & $f''(p)>0$, then f has a local min at p.
   - If $f'(p)=0$ & $f''(p)<0$, then f has a local max at p.
   - If $f'(p)=0$ & $f''(p)=0$, then tells nothing

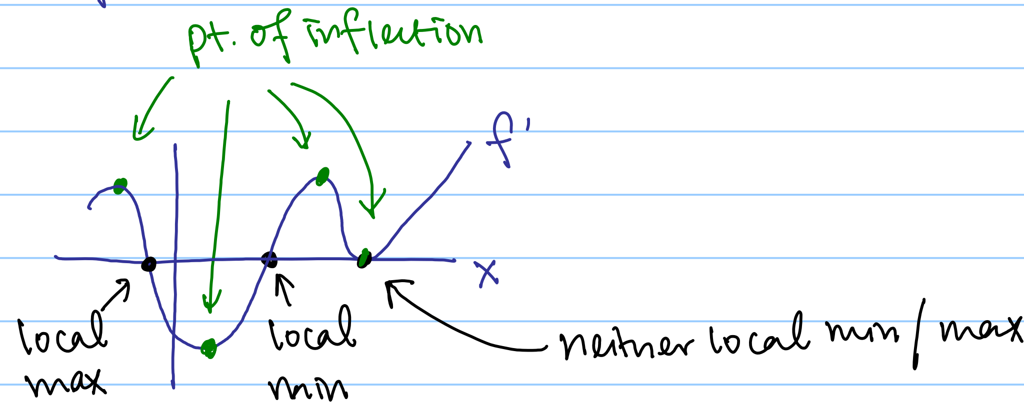
5. Inflection pt. : where f changes concavity.
   and point p where $f''(p)=0$ or undefined.
   - point p where f'' changes sign.





Q19 Indicate on the graph of derivative function f' the x-valued that are critical points. At which critical pt. does f has local min, local max or neither?

Q20 Indicate pt of inflection



4.2 Optimization

- f has a global min at p if $f(p) \leq$ all values of f
- f has a global max at p if $f(p) \geq$ all values of f

CHECK: critical pts + boundary pts!

Q7, 9 Find global min & global max on the closed interval.

7) $f(x) = xe^{-\frac{x^2}{2}}$ $\quad -2 \leq x \leq 2$

$$f'(x) = e^{-\frac{x^2}{2}} + xe^{-\frac{x^2}{2}}(-x)$$
$$= e^{-\frac{x^2}{2}}(1-x^2)$$

$f'(x) = 0$

$\Rightarrow \quad 1 - x^2 = 0 \qquad (\because e^{-\frac{x^2}{2}} \neq 0)$

$x = \pm 1$

Check: $f(1) = e^{-\frac{1}{2}} = 0.607$, $\quad f(-1) = e^{\frac{1}{2}} = -0.607$

$f(-2) = -0.271$, $\quad f(2) = 0.271$

$\therefore$ Global max value $= 0.607$ at $x = 1$

Global min value $= -0.607$ at $x = -1$

9) $f(x) = e^{-x}\sin x$ $\quad 0 \leq x \leq 2\pi$

$$f'(x) = -e^{-x}\sin x + e^{-x}\cos x = e^{-x}(-\sin x + \cos x)$$

$f'(x) = 0$

$\Rightarrow -\sin x + \cos x = 0$

$\sin x = \cos x \Rightarrow \tan x = 0$

i.e. $x = \frac{\pi}{4}$ or $\frac{5\pi}{4}$ in $0 \leq x \leq 2\pi$

$f(0) = 0$, $\quad f(2\pi) = 0$

$f(\frac{\pi}{4}) = 0.322$, $\quad f(\frac{5\pi}{4}) = -0.039$

$\therefore$ Global max value $= 0.322$ at $x = \frac{\pi}{4}$
Global min value $= -0.0139$ at $x = \frac{5\pi}{4}$

Q20,21 Find exact global min & max value. Domain $= \mathbb{R}$

20)

$$f(t) = \frac{t}{1+t^2}$$

$$f'(t) = \frac{1+t^2 - t(2t)}{(1+t^2)^2} = \frac{1-t^2}{(1+t^2)^2}$$

$$f'(t) = 0$$

$$\Rightarrow \quad 1 - t^2 = 0$$

$$t = \pm 1$$

	$t<-1$	$-1<t<1$	$t>1$
f'	$-$	$+$	$-$

$\therefore$ local min at $t = -1$, local max at $t = 1$

As $t \to \pm\infty$, $f(t) \to 0$

& $f(1) = \frac{1}{2}$, $f(-1) = -\frac{1}{2}$

$\therefore$ Global max value $= \frac{1}{2}$ at $t = 1$
Global min value $= -\frac{1}{2}$ at $t = -1$

21)

$$f(t) = (\sin^2 t + 2)\cos t$$

Note: $\sin^2 t + \cos^2 t = 1$

$$\therefore f(t) = (1 - \cos^2 t + 2)\cos t = (3 - \cos^2 t)\cos t$$

$$= 3\cos t - \cos^3 t$$

So $f'(t) = -3\sin t + 3\cos^2 t \sin t$

$$= 3\sin t(-1 + \cos^2 t) = 3\sin t(-\sin^2 t)$$

$$= -3\sin^3 t$$

$$f'(t) = 0$$
$$\Rightarrow \quad \sin^3 t = 0$$
$$\Rightarrow \quad t = n\pi \qquad n \text{ is an integer.}$$

Check: $f(n\pi) = (\sin^3(n\pi) + 2)\cos n\pi$
$$= \begin{cases} 2 & \text{if } n \text{ is even} \\ -2 & \text{if } n \text{ is odd} \end{cases}$$

$\therefore$ Global max value $= 2$ at $t = n\pi$ for n even
Global min value $= -2$ at $t = n\pi$ for n odd

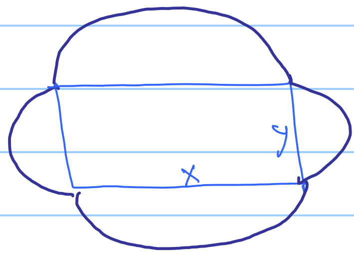
§ 4.4 Modeling

– word problem

Q3 An electric current I, in amps, is given by
$$I = \cos(wt) + \sqrt{3}\sin(wt) \quad \text{where } w \neq 0 \text{ is a constant.}$$
What are the max & min values of I?

Since $I(t)$ is periodic function with period $\frac{2\pi}{w}$, it is sufficient to consider $0 \le wt \le 2\pi$
$$\frac{dI}{dt} = -w\sin(wt) + \sqrt{3}w\cos(wt)$$
$$= w(\sqrt{3}\cos wt - \sin wt)$$
$$\frac{dI}{dt} = 0$$
$$\Rightarrow \quad \sqrt{3}\cos wt - \sin wt = 0$$
$$\tan wt = \sqrt{3}$$
$$\Rightarrow \quad wt = \frac{\pi}{3} \text{ or } \frac{4\pi}{3} \qquad \text{in } 0 \le wt \le 2\pi$$
$$I\left(\frac{\pi}{3}\right) = 2, \qquad I\left(\frac{4\pi}{3}\right) = -2$$
$\therefore$ max value $= 2$, min value $= -2$

Q.9



a) Find a formula of the area
b) Find a formula for the perimeter
c) Find x, y that maximize the area given perimeter = 100.

a) Area = $\square + \frown \times 2 + \supset + 2$

$$= xy + \frac{1}{2}\pi\left(\frac{x}{2}\right)^2 \times 2 + \frac{1}{2}\pi\left(\frac{y}{2}\right)^2 \times 2$$

$$= xy + \frac{\pi}{4}x^2 + \frac{\pi}{4}y^2$$

b) perimeter $= \pi x + \pi y$

c) $\pi x + \pi y = 100$

$$\Rightarrow \quad x = \frac{100 - \pi y}{\pi}$$

$$\text{Area} = A(y) = \left(\frac{100-\pi y}{\pi}\right)y + \frac{\pi}{4}\left(\frac{100-\pi y}{\pi}\right)^2 + \frac{\pi}{4}y^2$$

on $0 \le \pi y \le 100$ since y is a length.

$$A'(y) = \frac{100}{\pi} - 50 + (\pi - 2)y = 0$$

$$\Rightarrow \quad y = \frac{50}{\pi} = 15.916$$

Check $A(0) = A\left(\frac{100}{\pi}\right) = 795.8$, $A\left(\frac{50}{\pi}\right) = 651.2$

So Max at $y = 0,\ x = \frac{100}{\pi}$ or $y = \frac{100}{\pi},\ x = 0$ } max area = 795.8