CHAPTER 5. JET SPACES AND ARC SPACES

Contents

In this chapter we will discuss the basic theory of jet and arc spaces for schemes over $\mathbf{C}$ (or over any field of characteristic zero). There are by now many good surveys of the topic, especially from the point of view of motivic integration (see for instance [Bl], [Ve]) which will be our main focus in this course. The main sources I will use are [Mu] and [La].

Before getting into details, I would like to recall the p-adic inspiration for this theory. Here is a list of things we looked at in Chapter III.

(1) Polynomials $f \in \mathbf{Z}[X_1, \ldots, X_n]$ and their solutions over $\mathbf{Z}/p^{m+1}\mathbf{Z} \simeq \mathbf{Z}_p/p^{m+1}\mathbf{Z}_p$, where p is a fixed prime and $m \geq 0$. Such a solution can be written as

$$x_1 = a_{10} + a_{11}p + \ldots + a_{1m}p^m \, , \; \ldots \; , \; x_n = a_{n0} + a_{n1}p + \ldots + a_{nm}p^m$$

with $a_i \in \{0, \ldots, p-1\}$.

(2) Solutions of f over $\mathbf{Z}_p = \varprojlim_m \mathbf{Z}/p^{m+1}\mathbf{Z}$.

(3) The p-adic norm $|f(x)|_p = \frac{1}{p^{\mathrm{ord}_p f(x)}}$.

(4) p-adic integrals on $\mathbf{Z}_p^n$, with respect to the Haar measure $d\mu$.

We would now like to look at polynomials $f \in \mathbf{C}[X_1, \ldots, X_n]$. The point is to replace by analogy formal power series in the p-adic setting with the usual formal power series over $\mathbf{C}$. More precisely, we will look at:

(1') Polynomials $f \in \mathbf{C}[X_1, \ldots, X_n]$ and their solutions over $\mathbf{C}[t]/(t^{m+1}) \simeq \mathbf{C}[[t]]/(t^{m+1})$, with $m \geq 0$. Such a solution can be written as

$$x_1 = a_{10} + a_{11}t + \ldots + a_{1m}t^m \, , \; \ldots \; , \; x_n = a_{n0} + a_{n1}t + \ldots + a_{nm}t^m$$

with $a_i \in \mathbf{C}$, i.e. an *m-th jet* of $f = 0$.

(2') Solutions of f over $\mathbf{C}[[t]] = \varprojlim_m \mathbf{C}[[t]]/(t^{m+1})$. Such a solution is a collection of formal power series $\gamma = (\gamma_1, \ldots, \gamma_n)$, called an arc of $f = 0$.

(3') A norm $|f(\gamma)|_{\mathbf{C}((t))} = \frac{1}{q^{\mathrm{ord}_\gamma f}}$, where $q > 1$ is any fixed real numeber.

(4') Integrals over the arc space $\mathbf{A}^n_\infty = \mathrm{Spec}(\mathbf{C}[\![t]\!])^n$. This last step should be useful, like in the p-adic case, but it is a priori unclear how to do it since there are no known good $\mathbf{R}$-valued measures on $\mathbf{A}^n_\infty$. Following Kontsevich, we will need to introduce a measure with values, roughly speaking, in the Grothendieck ring of varieties.

1. Jet spaces

Given a non-negative integer m, we use the notation

$$\Delta_m = \mathrm{Spec}\ \mathbf{C}[t]/(t^{m+1}) \quad \text{and} \quad \Delta = \mathrm{Spec}\ \mathbf{C}[\![t]\!].$$[1]

There are obvious inclusion maps

$$\Delta_0 \subset \Delta_1 \subset \cdots \subset \Delta_m \subset \cdots \subset \Delta. \tag{1}$$

Definition 1.1. Let X be a scheme over $\mathbf{C}$. Set theoretically, the *space of m-th order jets* on X consists of the morphisms of schemes over $\mathbf{C}$

$$X_m := \mathrm{Hom}(\Delta_m, X),$$

while the *space of arcs* is

$$X_\infty := \mathrm{Hom}(\Delta, X).$$

The inclusions in (1) give rise to *truncation maps*

$$\pi_m : X_\infty \longrightarrow X_m$$

and

$$\pi^k_m : X_k \longrightarrow X_m, \quad k \geq m.$$

These spaces have natural scheme structures. To see this in a rather elementary way, it is important to understand first the situation for affine space and then affine schemes.

Lemma 1.2. *We have $(\mathbf{A}^n)_m = \mathbf{A}^{n(m+1)}$ and for each $k \geq m$, the truncation map*

$$\pi^k_m : (\mathbf{A}^n)_k \longrightarrow (\mathbf{A}^n)_m$$

is given (up to permutation) by projection onto the first $n(m+1)$ coordinates.

Proof. An element in $(\mathbf{A}^n)_m$ is given by a $\mathbf{C}$-algebra homomorphism

$$\varphi : \mathbf{C}[X_1, \ldots, X_n] \longrightarrow \mathbf{C}[t]/(t^{m+1}).$$

This is described by the images of the variables $u_i = \varphi(X_i)$. Each u_i can be written as

$$u_i = u_{i0} + u_{i1}t + \ldots + u_{im}t^m,$$

[1]This is meant to symbolize that Δ is playing the role of a disk in the usual topology; it is the "formal disk".

with no conditions on the u_{ij}. Thinking of $(u_{i0}, \ldots, u_{im})$, $i = 1, \ldots, n$, as coordinate vectors, this gives the stated isomorphism. The second statement is clear, since for the truncation map we would be factoring φ through a homomorphism to $\mathbf{C}[t]/(t^{k+1})$. $\square$

Let now $X \subset \mathbf{A}^n$ be an arbitrary affine scheme, given by an ideal $I \subset \mathbf{C}[X_1, \ldots, X_n]$. Assume that I is generated by $f_1, \ldots, f_r$. An m-th jet on X is given by a homomorphism

$$A(X) = \mathbf{C}[X_1, \ldots, X_n]/I \longrightarrow \mathbf{C}[t]/(t^{m+1}).$$

If we write a jet as a vector $(u_1, \ldots, u_n)$ as above, we can then see X_m as being the affine subscheme of $(\mathbf{A}^n)_m$ given by the equations

$$f_i(u_1, \ldots, u_n) = 0, \quad i = 1, \ldots, r.$$

Exercise 1.3. Check that the scheme structure on X_m defined above is independent of the generators f_i and on the affine embedding $X \subset \mathbf{A}^n$. Conclude that for an arbitrary scheme X we obtain a well-defined scheme structure on X_m. Check that for every open subset $U \subset X$ we have $U_m = (\pi_0^m)^{-1}(U)$ (in fact start the whole thing by doing this for X affine).

There is a more functorial way of recognizing the natural scheme structure on X_m (and in particular get the conclusion of the Exercise above), by means of the fact that X_m represents a functor. For the general theory of the functor of points of a scheme, and how it determines it, see e.g. [Mu] Ch.II §6. Concretely, consider the functor

$$J_m : \{\text{Schemes over } \mathbf{C}\} \longrightarrow \{\text{Sets}\}, \quad J_m(S) = \text{Hom}(S \times \Delta_m, X).$$

Proposition 1.4. *The functor J_m is represented by X_m, i.e. for each $\mathbf{C}$-scheme S we have functorial bijections*

$$\text{Hom}(S \times \Delta_m, X) \simeq \text{Hom}(S, X_m).$$

Proof. It suffices to show this for affine schemes, in other words to show that for all $\mathbf{C}$-algebras A we have functorial bijections

$$\text{Hom}(\text{Spec } A[t]/(t^{m+1}), X) \simeq \text{Hom}(\text{Spec } A, X_m).$$

If X is affine, this follows precisely as in the argument above. If X is arbitrary, we cover it by open affines $U_1, \ldots, U_r$, so that we do have the conclusion we want for $(U_i)_m$ for all i. As in the Exercise above, both $(\pi_0^{m,i})^{-1}(U_i \cap U_j)$ and $(\pi_0^{m,j})^{-1}(U_i \cap U_j)$ are isomorphic over $U_i \cap U_j$ with $(U_i \cap U_j)_m$. So the scheme X_m can be constructed by gluing $(U_i)_m$ over these overlaps, and it clearly satisfies the property we want. $\square$

Example 1.5. Note that by definition, when X is a smooth variety, the $\mathbf{C}$-valued points of the space of first-order jets form the total space of the tangent bundle T_X. More generally, given a scheme X, the first jet-scheme X_1 is the total space of the tangent sheaf, i.e. $X_1 = \mathbf{Spec}(\text{Sym } \Omega_X^1)$. It is enough to see this for affine schemes $X = \text{Spec } R$. We use the interpretation in Proposition 1.4: given any $\mathbf{C}$-algebra A, note that giving a morphism

$$\text{Spec } A \longrightarrow \mathbf{Spec}(\text{Sym } \Omega_X^1)$$

is equivalent to giving a morphism of **C**-algebras $f : R \to A$ and a **C**-derivation $D : R \to A$. This in turn is the same as specifying a homomorphism

$$g : R \longrightarrow A[t]/(t^2), \quad \text{with } g(r) = f(r) + tD(r).$$

Example 1.6. Let's describe all X_m for $X = (xy = 0) \subset \mathbf{A}^2$. We are looking for solutions of the form

$$x = x_0 + x_1 t + \ldots + x_m t^m \quad \text{and} \quad y = y_0 + y_1 t + \ldots + y_m t^m$$

such that $xy = 0 \pmod{t^{m+1}}$. (Recall that one such gives a point $(x_0, \ldots, x_m, y_0, \ldots, y_m) \in \mathbf{A}^{2(m+1)} = (\mathbf{A}^2)_m$.) They can be produced as follows: consider for any integers $k, l \geq 0$ such that $k + l = m + 1$, the subsets

$$V_{k,l} = \{x_0 = \ldots = x_{k-1} = y_0 = \ldots = y_{l-1} = 0\} \subset \mathbf{A}^{2(m+1)}.$$

Clearly $V_{k,l} \simeq \mathbf{A}^{m+1}$. We see that $V_{k,l}$ parametrizes jets of the form

$$(x, y) = (x_k t^k + x^{k+1} t^{k+1} + \ldots, \; y_l t^l + y^{l+1} t^{l+1} + \ldots)$$

and all solutions must belong to one of them. The $V_{k,l}$ are in fact precisely the irreducible components of X_m. Note that under the truncation map

$$\pi_0^m : X_m \longrightarrow X$$

$V_{m+1,0}$ and $V_{0,m+1}$ map to the x and y axes respectively, while all the other $V_{k,l}$ map to the origin (which is the singularity of X).

Example 1.7. Let $X = (xy - z^3 = 0) \subset \mathbf{C}^3$. We will show that for each $m \geq 1$, X_m is irreducible of dimension $2(m + 1)$. For a fixed m, we are looking for

$$x(t) = a_0 + a_1 t + \ldots + a_m t^m, \; y(t) = b_0 + b_1 t + \ldots + b_m t^m, \; z(t) = c_0 + c_1 t + \ldots + c_m t^m$$

such that

$$x(t) \cdot y(t) = z(t)^3 \bmod t^{m+1}. \tag{2}$$

Note that the identity (2) consists of $m + 1$ equations in the $a's$, $b's$ and $c's$, i.e. in the affine space $(\mathbf{A}^3)_m = \mathbf{A}^{3(m+1)}$. This implies that each irreducible component of X_m has dimension at least $2(m + 1)$. We now have to distinguish between those irreducible components of X_m that do not lie over the origin, and those that do. We will show that there is precisely one of the former, and none of the latter.

Let $Y \subset X_m$ be the union of components that do not lie over the origin. It is easy to see that for the general $(x(t), y(t), z(t)) \in Y$ we have $a_0, b_0, c_0 \neq 0$. Now if $a_0 \neq 0$, x can be inverted, so that we can solve for y in terms of x and z. In other words, if the a's and c's are general, one can uniquely solve for the b's. This implies that Y is irreducible, of dimension precisely $2(m + 1)$.

On the other hand, I claim that

$$\dim\{(x(t), y(t), z(t)) \mid a_0 = b_0 = c_0 = 0\} < 2(m + 1),$$

which immediately implies that there are no extra components lying over the origin. Indeed, if all the initial terms are 0, after factoring out (at least) one power of t in x, y and z, we can repeat the same process and obtain loci of dimension at most $2m$.

Exercise 1.8. (1) Let $X = (x^3+y^3+z^3 = 0) \subset \mathbf{C}^3$. Show that for every $m \geq 1$, X_m has a unique irreducible component which dominates X via the truncation map $\pi_0^m : X_m \to X$. This component has dimension $2(m+1)$. Show that for $3|m+1$, X_m has extra irreducible components lying over the origin in X, which are of dimension $2(m+1)$ as well.

(2) Let $X = (x^d + y^d + z^d = 0) \subset \mathbf{C}^3$, with $d \geq 4$. Show that for every $m \geq 1$, X_m has a unique irreducible component which dominates X via the truncation map $\pi_0^m : X_m \to X$. This component has dimension $2(m+1)$. Show that for $d|m+1$, X_m has extra irreducible components lying over the origin in X, which this time have dimension $> 2(m+1)$.

Definition 1.9. We say that a morphism $f : X \to Y$ is *locally trivial with fiber F* if there exists a Zariski open cover $Y = U_1 \cup \ldots \cup U_k$ such that $f^{-1}(U_i) \simeq U_i \times F$ for all i, with the restriction of f to $f^{-1}(U_i)$ being the projection onto the first component. If this condition holds for a cover where U_i are only locally closed, we call f *piecewise trivial with fiber F*.

Theorem 1.10. *Let X be a smooth complex variety of dimension n. Then, for each $m \geq 1$, X_m is smooth of dimension $n(m+1)$. Moreover, the truncation maps*

$$\pi_m^{m+e} : X_{m+e} \longrightarrow X_m, \quad \text{with } e > 0,$$

are locally trivial in the Zariski topology, with fiber $\mathbf{A}^{ne}$.

Proof. It is clear that the local-triviality statement implies the smoothness and dimension conclusions as well. For this statement, it is enough to assume that $e = 1$, since then we can proceed inductively.

To this end, since we know that for an open set $U \subset X$ we have $(\pi_0^m)^{-1}(U) = U_m$, it suffices to prove the following: for any open set $U \subset X$ having an algebraic coordinate system $x_1, \ldots, x_n$, we have an isomorphism $U_m \simeq U \times \mathbf{A}^{mn}$ such that π_m^{m+1} is the projection onto the first mn components. The condition of being an algebraic coordinate system means that $x_1, \ldots, x_n \in \mathcal{O}_X(U)$ are such that the differentials $dx_1, \ldots, dx_n$ trivialize Ω_X^1 over U; such a system exists around every point since X is smooth.

On the other hand, an algebraic coordinate system is equivalent to the data of an étale morphism $U \to \mathbf{A}^n$. By Lemma 1.11 below and Lemma 1.2, we conclude that

$$U_m \simeq (\mathbf{A}^n)_m \times_{\mathbf{A}^n} U = \mathbf{A}^{n(m+1)} \times_{\mathbf{A}^n} U \simeq \mathbf{A}^{nm} \times U,$$

and the assertion about the projection map is clear. □

Note that by functoriality each morphism of schemes $f : X \to Y$ induces morphisms $f_m : X_m \to Y_m$, compatible with the truncation maps.

Lemma 1.11. *Let $f : X \to Y$ be an étale morphism of schemes. Then, for every $m \geq 1$, the induced commutative diagram*

$$\begin{array}{ccc} X_m & \xrightarrow{f_m} & Y_m \\ \downarrow{\scriptstyle \pi_0^m} & & \downarrow{\scriptstyle \pi_0^m} \\ X & \xrightarrow{f} & Y \end{array}$$

is cartesian (i.e a fiber product).

Proof. Using the interpretation of X_m and Y_m as representing the respective functors J_m as in Proposition 1.4, it is enough to show that for every $\mathbf{C}$-algebra A and every commutative diagram

$$\begin{array}{ccc} \text{Spec } A & \longrightarrow & X \\ \downarrow & & \downarrow f \\ \text{Spec } A[t]/(t^{m+1}) & \longrightarrow & Y \end{array}$$

there is a unique morphism Spec $A[t]/(t^{m+1}) \to X$ of schemes over $\mathbf{C}$ making the two induced triangles commutative. This is a consequence of the commutative algebra fact that étale morpisms are formally étale (see e.g. [Ma] §28). □

Exercise 1.12. Show that for an arbitrary scheme X, the truncation maps $\pi_m^{m+1} : X_{m+1} \to X_m$ are affine morphisms.

Exercise 1.13. Show that if C is a singular curve, then C_1 is not irreducible.

2. Arc spaces

Generalities. Given a scheme X, we have constructed jet-schemes X_m for each $m \geq 0$, together with truncation maps $\pi_m^{m+1} : X_{m+1} \to X_m$, which are affine morphisms; in other words, for every open affine subset $U \subset X_m$, $(\pi_m^{m+1})^{-1}(U)$ is affine, and the corresponding restriction of the truncation map corresponds to a ring homomorphism with arrows reversed. Since inductive limits exist in the category of $\mathbf{C}$-algebras, this implies that we can pass to the projective limit in the category of schemes over $\mathbf{C}$ to obtain:

Definition 2.1. The (scheme-theoretic) *arc space* of X is

$$X_\infty := \varprojlim_m X_m.$$

Even if X is a scheme of finite type over $\mathbf{C}$, in general X_∞ will not be of finite type. Note that by definition there exist *projection morphisms*

$$\psi_m : X_\infty \longrightarrow X_m \quad \text{for all } m \geq 0,$$

and by the definition of the projective limit scheme structure on X_∞, for every open set $U \subset X$ we have

$$\mathcal{O}_{X_\infty}(\psi_0^{-1}(U)) \simeq \varinjlim_m \mathcal{O}_{X_m}((\pi_0^m)^{-1}(U)).$$

The Zariski topology on X_∞ is the projective limit topology; in other words, the closed subsets of X_∞ are limits of compatible closed subsets in $Z_m \subset X_m$, i.e. satisfying

$$\pi_m^{m+1}(Z_{m+1}) = Z_m \quad \text{for } m \gg 0.$$

For example, given any closed subscheme $Z \subset X$, Z_∞ is a closed subset of X_∞.

For every $\mathbf{C}$-algebra A, using Proposition 1.4 we have

$$\text{Hom}(\text{Spec } A, X_\infty) \simeq \varprojlim_m \text{Hom}(\text{Spec } A, X_m) \simeq$$

$$\simeq \varprojlim_m \mathrm{Hom}(\mathrm{Spec}\ A[t]/(t^{m+1}), X) \simeq \mathrm{Hom}(\mathrm{Spec}\ A[\![t]\!], X).$$

In particular, the $\mathbf{C}$-valued points of X_∞ are precisely the set-theoretic space of arcs $\mathrm{Hom}(\Delta, X)$ defined in the previous section.

Exercise 2.2. (1) Show that if $f : X \to Y$ is a morphism of schemes, there is an induced natural morphism $f_\infty : X_\infty \to Y_\infty$, commuting with all the projection morphisms ψ_m. If f is a closed immersion, show that f_∞ is a closed immersion.

(2) If f is étale, show that the commutative diagram

$$\begin{array}{ccc} X_\infty & \xrightarrow{f_\infty} & Y_\infty \\ \downarrow{\scriptstyle \psi_0} & & \downarrow{\scriptstyle \psi_0} \\ X & \xrightarrow{f} & Y \end{array}$$

is a fiber product.

Equations for jet and arc spaces. Here I will note that there is an algorithmic procedure for producing equations for X_m and X_∞ starting with equations for an affine $X \subset \mathbf{A}^n$. Starting with the polynomial ring $S = \mathbf{C}[X_1, \ldots X_n]$, we formally introduce new variables $X_i^{(m)}$ for each $m \geq 1$, with $X_i^{(0)} = X_i$. Define

$$S_\infty = \mathbf{C}[\ X_i^{(m)} \mid i = 1, \ldots, n,\ m \geq 0\ ].$$

We have $(\mathbf{A}^n)_\infty = \mathrm{Spec}\ S_\infty$. We can define a $\mathbf{C}$-derivation on S_∞ by the rule

$$D : S_\infty \longrightarrow S_\infty,\quad D(X_i^{(m)}) = X_i^{(m+1)}.$$ [2]

Given a polynomial $f \in S$, letting $f^{(0)} = f$, we can define recursively $f^{(m)} = D(f^{(m-1)})$.

Let now $X \subset \mathbf{A}^n$ be given by the ideal $I = (f_1, \ldots, f_r) \subset S$, and let $R = A(X) = S/I$. Define

$$R_\infty := S_\infty / I_\infty,\quad I_\infty = (f_i, f_i^{(1)}, \ldots, f_i^{(m)}, \ldots),\ i = 1, \ldots, r.$$

Lemma 2.3. $X_\infty \simeq \mathrm{Spec}\ R_\infty$.

Proof. For any $\mathbf{C}$-algebra A, a $\mathbf{C}$-homomorphism

$$\varphi : \mathbf{C}[X_1, \ldots, X_n] \longrightarrow A[\![t]\!]$$

is defined by the images of the variables

$$\varphi(X_i) = \sum_{m \geq 0} \frac{a_i^{(m)}}{m!} \cdot t^m.$$

This gives after a small calculation that for every $f \in \mathbf{C}[X_1, \ldots, X_n]$ we have

$$\varphi(f) = \sum_{m \geq 0} \frac{f^{(m)}(a, a^{(1)}, \ldots, a^{(m)})}{m!} \cdot t^m,$$

[2] The basic thinking is that we are introducing a new variable $X_i^{(m)}$ for each differential operator given by the partial derivative $\frac{\partial^m}{\partial X_i^m}$.

where $a = (a_1, \ldots, a_r)$ and similarly for $a^{(m)}$. To say that φ induces a homomorphism $R \to A[\![t]\!]$ is then equivalent to saying that

$$f_i^{(m)}(a, a^{(1)}, \ldots, a^{(m)}) = 0 \quad \text{for all } m \geq 0 \quad \text{and all } 1 \leq i \leq r.$$

□

By truncating the above discussion at level m, we obtain of course equations for the jet-scheme X_m. In other words, we can consider

$$S_m = \mathbf{C}[\ X_i^{(j)} \mid i = 1, \ldots, n,\ j = 0, \ldots, m\ ]$$

and

$$R_m := S_m/(f_i, f_i^{(1)}, \ldots, f_i^{(m)} \mid i = 1, \ldots, r).$$

The conclusion is that $X_m \simeq \text{Spec } R_m$, and the truncation maps are obtained from the natural morphisms $R_m \to R_{m+1}$.

Example 2.4. Let X be the cusp $y^2 - x^3 = 0$ in $\mathbf{A}^2$. Then X_1 is defined in Spec $\mathbf{C}[X, Y, X', Y']$ by the ideal

$$(Y^2 - X^3, 2YY' - 3X^2X')$$

while X_2 is defined in Spec $\mathbf{C}[X, Y, X', Y', X'', Y'']$ by the ideal

$$(Y^2 - X^3, 2YY' - 3X^2X', 2Y'^2 + 2YY' - 6XX'^2 - 3X^2X'').$$

Kolchin's theorem. Before proving the next result, let's observe the following. An arc $\gamma : \text{Spec } \mathbf{C}[\![t]\!] \to X$ determines two points on X, corresponding to the two points of Spec $\mathbf{C}[\![t]\!] \to X$; one is the image $\gamma\big((t)\big)$ of its closed point, while the other is the image $\gamma\big((0)\big)$ of its generic point. The latter corresponds to the induced morphism

$$\bar{\gamma} : \text{Spec } \mathbf{C}((t)) \longrightarrow X.$$

Proposition 2.5. *Let $f : Y \to X$ be a proper birational morphism of varieties. If $Z \subset X$ is a proper closed subset such that f is an isomorphism over $X - Z$, then the induced map*

$$Y_\infty - f^{-1}(Z)_\infty \longrightarrow X_\infty - Z_\infty$$

is a bijection.

Proof. Let $\gamma \in X_\infty$. Then $\gamma \notin Z_\infty$ if and only if the induced $\bar{\gamma} : \text{Spec } \mathbf{C}((t)) \longrightarrow X$ has its image in $U = X - Z$. In this case $\bar{\gamma}$ has a lifting to a morphism

$$\text{Spec } \mathbf{C}((t)) \longrightarrow Y - f^{-1}(Z),$$

since f is an isomorphism over U. We now apply the Valuative Criterion of Properness (see [Ha], Ch.II Theorem 4.7) for such an arc. Note that (identifying $\bar{\gamma}$ with its lifting) we have a commutative diagram

$$\begin{array}{ccc} \text{Spec } \mathbf{C}((t)) & \xrightarrow{\bar{\gamma}} & Y \\ \downarrow & & \downarrow f \\ \text{Spec } \mathbf{C}[\![t]\!] & \xrightarrow{\gamma} & X \end{array}$$

The valuative criterion says then that there exists a unique lifting of γ to an arc Spec $\mathbf{C}[\![t]\!] \to Y$, which necessarily has to be in $Y_\infty - f^{-1}(Z)_\infty$. $\square$

Theorem 2.6 (Kolchin)**.** *If X is a complex variety, then X_∞ is irreducible.*

Proof. We prove this only for the complex points of X_∞ (but the general statement holds as well). First, when X is smooth, we have seen that all X_m are smooth varieties. Therefore passing to the limit $X_\infty = \varprojlim_m X_m$ we obtain an irreducible scheme as well.

Let's assume now that X is singular, and prove the result by induction on $n = \dim X$. Consider a resolution of singularities $f : Y \to X$, and let $Z \subset X$ be a proper closed subset such that f is an isomorphism over $X - Z$. Proposition 2.5 implies that

$$X_\infty = Z_\infty \cup \mathrm{Im}(f_\infty).$$

Now since Y is smooth, by the above we know that Y_∞ is irreducible, and therefore $\mathrm{Im}(f_\infty)$ is irreducible as well. To conclude, it suffices then to show that $Z_\infty \subset \overline{\mathrm{Im}(f_\infty)}$.

Consider now the decomposition of Z into its irreducible components, $Z = Z_1 \cup \ldots \cup Z_k$. This gives

$$Z_\infty = Z_{1\infty} \cup \ldots Z_{k\infty}.$$

For every $i = 1, \ldots, k$, pick an irreducible component Y_i of $f^{-1}(Z_i)$ such that the induced $Y_i \to Z_i$ is surjective. By generic smoothness (see [Ha], Ch.III Corollary 10.7), there exist open subsets $U_i \subset Y_i$ and $V_i \subset Z_i$ such that f restricts to a surjective smooth morphism $g_i : U_i \to V_i$. This implies that

$$V_{i\infty} = \mathrm{Im}(g_{i\infty}) \subset \mathrm{Im}(f_\infty).$$

Now by induction on dimension we know that all $Z_{i\infty}$ are irreducible. They contain $V_{i\infty}$ as non-empty open sets, so passing to closures we conclude that $Z_{i\infty} \subset \overline{\mathrm{Im}(f_\infty)}$. $\square$

3. Cylinders and the birational transformation rule

Let X be a smooth complex variety of dimension n. While X_∞ is infinite dimensional, it often suffices to restrict one's attention to subsets which come from a finite level, i.e. from nice enough subsets of some X_m.

Definition 3.1. (1) Let Y be a variety. A *constructible subset* of Y is a finite union of locally closed subsets of Y.

(2) A *cylinder* in X_∞ is a set of the form $\psi_m^{-1}(C_m)$ for some m, where $C_m \subset X_m$ is a constructible set. A cylinder C is open, closed, locally closed or irreducible if it can be written as above, with C_m having the respective property. Cylinders are closed under finite unions, intersections, and complements, and therefore they form an algebra of sets.

Example 3.2. The image of the morphism $f : \mathbf{A}^2 \to \mathbf{A}^2$, $f(x,y) = (x, xy)$ (i.e. the restriction of the blow-up of $\mathbf{A}^2$ at the origin to one of its charts) is constructible, but not locally closed around the origin. The image of any morphism of finite type is constructible, according to Chevalley's theorem.

Example 3.3. (1) The most interesting cylinders we will discuss arise from looking at the vanishing orders of arcs. Fix a proper closed subscheme $Z \subset X$. One defines a function

$$\mathrm{ord}_Z : X_\infty \longrightarrow \mathbf{N} \cup \infty, \quad \gamma \mapsto \mathrm{ord}_Z(\gamma)$$

given by the vanishing order of γ along Z: explicitly, if $\gamma : \mathrm{Spec}\ \mathbf{C}[\![t]\!] \to X$, then the scheme theoretic preimage of Z is defined by an ideal in $\mathbf{C}[\![t]\!]$, generated by $t^{\mathrm{ord}_Z(\gamma)}$ (note that every ideal in $\mathbf{C}[\![t]\!]$ is generated by some power of t). We define the *m-th contact locus* of Z to be

$$\mathrm{Cont}^m(Z) := \mathrm{ord}_Z^{-1}(m) \subset X_\infty.$$

Similarly, we can define $\mathrm{Cont}^{\geq m}(Z) := \mathrm{ord}_Z^{-1}(\geq m)$, and we note that

$$\mathrm{Cont}^{\geq m}(Z) = \psi_{m-1}^{-1}(Z_{m-1}).$$

(Indeed, it is enough to assume that $X = \mathrm{Spec}\ R$, with Z being given by an ideal $I \subset R$. An arc $\gamma \in \mathrm{Cont}^{\geq m}(Z)$ is given by a homomorphism $\varphi : R \to \mathbf{C}[\![t]\!]$ such that $\varphi(I) = (t^k)$ with $k \geq m$. This implies that I is mapped to 0 by composition with the truncation map $\mathbf{C}[\![t]\!] \to \mathbf{C}[\![t]\!]/(t^m)$, so that finally this data is equivalent to a homomorphism $R/I \to \mathbf{C}[\![t]\!]/(t^m)$, or in other words to an arc in Z_{m-1}.) This implies that $\mathrm{Cont}^{\geq m}(Z)$ is a closed cylinder, and consequently $\mathrm{Cont}^m(Z)$ is a locally closed cylinder.

(2) The basic non-example is the following: if $Z \subset X$ is a proper subscheme, then $Z_\infty \subset X_\infty$ is not a cylinder. This amounts to the fact that one cannot test membership in Z_∞ by looking at jets of some fixed order m.

Exercise 3.4. Let $f : X \to Y$ be a proper birational morphism of smooth varieties. Show that:

(1) $f_m : X_m \to Y_m$ is surjective for every $m \geq 0$.

(2) If $F \subset X_m$ is a union of fibers of f_m and $C = \psi_m^{-1}(F)$, then

$$f_\infty(C) = \psi_m^{-1}(f_m(F)).$$

In particular, $f_\infty(C)$ is a cylinder.

(3) $f_\infty : X_\infty \to Y_\infty$ is surjective.

Let now $f : X \to Y$ be a proper birational morphism of smooth varieties of dimension n. We are particularly interested in the contact loci defined by the relative canonical divisor

$$C^p := \mathrm{Cont}^p(K_{X/Y}).$$

The behavior of these loci is explained by the following celebrated Birational Transformation Theorem. This will be the key ingredient for the Change of Variables Theorem for motivic integration, discussed in the next chapter.

Theorem 3.5 (Kontsevich)**.** *With the notation above, let m and e be integers such that $m \geq 2e$. Then:*

(i) For every $\gamma, \gamma' \in X_m$ such that $\gamma \in \psi_m(C^e)$ and $f_m(\gamma) = f_m(\gamma') \in Y_m$, we have

$$\pi^m_{m-e}(\gamma) = \pi^m_{m-e}(\gamma').$$

(In other words, the fiber of f_m through a point in $\psi_m(C^e)$ is contained in a fiber of π^m_{m-e}.)

(ii) $\psi_m(C^e)$ is a union of fibers of f_m, each isomorphic to $\mathbf{A}^e$.

(iii) The induced map

$$\psi_m(C^e) \longrightarrow f_m\big(\psi_m(C^e)\big)$$

is piecewise trivial with fiber $\mathbf{A}^e$.

Corollary 3.6. *In the notation of the Theorem, $f_\infty(C^e)$ is a cylinder.*

Proof. This follows from (ii) in the Theorem and Exercise 3.4. □

Example 3.7. Let

$$f : \mathbf{A}^2 \to \mathbf{A}^2, \quad f(x,y) = (x, xy)$$

be the restriction of the blow-up of the origin of $\mathbf{A}^2$ to one of its two charts. The Jacobian of f is x, and therefore the relative canonical divisor is $K = (x = 0)$. Consider now a jet

$$\gamma = (x_0 + x_1 t + \ldots + x_m t^m, y_0 + y_1 t + \ldots + y_m t^m)$$

in $(\mathbf{A}^2)_m$. The condition $\gamma = \psi_m(C^e)$ (i.e. of γ having order of contact with K precisely e) translates into $x_0 = x_1 = \ldots = x_{e-1} = 0$ and $x_e \neq 0$. We have

$$f_m(\gamma) = \big(x_e t^e + \ldots + x_m t^m, (x_e t^e + \ldots + x_m t^m)(y_0 + y_1 t + \ldots + y_m t^m) \bmod t^{m+1}\big)\,.$$

As a consequence

$$f_m^{-1}(f_m(\gamma)) = \{\gamma + (0, z_{m-e+1} t^{m-e+1} + \ldots + z_m t^m)\}$$

with z_j arbitrary, since nothing beyond order $m-e$ in the y's can influence the expression for $f_m(\gamma)$. This clearly implies that $f_m^{-1}(f_m(\gamma)) \simeq \mathbf{A}^e$, and for every $\gamma' \in f_m^{-1}(f_m(\gamma))$ we have $\pi^m_{m-e}(\gamma) = \pi^m_{m-e}(\gamma')$.

Remark 3.8. Let $f : X \to Y$ be a proper birational morphism of smooth varieties, and let $Z \subset Y$ be the image of the exceptional locus of f. Since Z_∞ consists precisely of those arcs having infinite contact order with Z, and same for $f^{-1}(Z)$, we have

$$X_\infty - f^{-1}(Z)_\infty = \coprod_{e \in \mathbf{N}} C^e \quad \text{and} \quad Y_\infty - Z_\infty = \coprod_{e \in \mathbf{N}} f_\infty(C^e).$$

Note that f_∞ induces a bijection on these loci, and in particular the mapping $C^e \to f_\infty(C^e)$ is bijective for each e. However, Theorem 3.5 says that when truncating to finite level $m \geq 2e$, the induced mapping in piecewise trivial with fiber $\mathbf{A}^e$.

The rest of this section is concerned with the proof of Theorem 3.5. Let's first note that the statement in part (iii) follows from the previous statements and the following general

Lemma 3.9. *Let $f : V \to W$ be a morphism of schemes of finite type such that the fibers over all (not necessarily closed) points of W are isomorphic as schemes to $\mathbf{A}^e$. Then f is a piecewise trivial fibration with fiber $\mathbf{A}^e$.*

Proof. We can assume that W is irreducible. The fiber of f over the generic point $\eta \in W$ is isomorphic to $\mathbf{A}^e$, which means that there exists an open set $U \subset W$ such that $f^{-1}(U) \simeq U \times \mathbf{A}^e$. But now the induced morphism $V - f^{-1}(U) \to W - U$ is a morphism between schemes of finite type of strictly lower dimension than before, and we can conclude by induction. □

On the other hand, the fact that $\psi_m(C^e)$ is a union of fibers of f_m follows from (i) due to the fact that C^e is stable at level e in the sense of the following

Definition 3.10. A subset $S \subset X_\infty$ is *stable at level* k if for any $m \geq k$ we have that $\psi_m(S) \subset X_m$ is constructible, $\psi_m^{-1}(\psi_m(S)) = S$, and the induced truncation map

$$\pi_m^{m+1} : \psi_{m+1}(S) \longrightarrow \psi_m(S)$$

is a locally trivial fibration with fiber $\mathbf{A}^n$. S is called *stable* if it is stable at some level.

Indeed, it is clear that C^e is stable at level e, since the condition of an arc being in C^e depends only on its truncation to level e. Since $m \geq 2e$, we then have that $\psi_m(C^e) \to \psi_{m-e}(C^e)$ is locally trivial with fiber $\mathbf{A}^{ne}$, which is the entire fiber of π_{m-e}^m. This, combined with (i), immediately implies the statement in (ii).

In conclusion, for the Theorem one only needs to check that each fiber $f_m^{-1}(f_m(\gamma))$ with $\gamma \in \psi_m(C^e)$ is contained in a fiber of π_{m-e}^m, and is isomorphic to $\mathbf{A}^e$.

The case of affine space. Since the ideas are more transparent in this case, for intuition let's first treat the case $X = Y = \mathbf{A}^n$. Let

$$f = (f_1, \ldots, f_n) : \mathbf{A}^n \longrightarrow \mathbf{A}^n, \quad f_i \in \mathbf{C}[X_1, \ldots, X_n].$$

Let

$$\gamma' \in X_\infty = \mathbf{A}^n_\infty,$$

i.e. a collection of n formal power series in the variable t. Assume that $\gamma' \in C^e$. Write

$$\psi_m(\gamma') = \gamma = (\gamma_1, \ldots, \gamma_n),$$

so that γ_i are polynomials of degree m in t, with $\mathrm{ord}_{K_{X/Y}}(\gamma) = e$. In order to describe the fiber

$$f_m^{-1}(f_m(\gamma))$$

we need to describe all n-tuples $\beta = (\beta_1, \ldots, \beta_n)$ of polynomials of degree m in t satisfying

$$f(\gamma + \beta) \equiv f(\gamma) \bmod t^{m+1}.$$

Since f is given by polynomials, we can expand using Taylor's formula, to get

$$f(\gamma + \beta) = f(\gamma) + Df(\gamma) \cdot \beta + \text{ higher order terms in } \beta, \tag{3}$$

where we have

$$Df(\gamma) = \left(\frac{\partial f_i}{\partial X_j} (\gamma_1(t), \ldots, \gamma_n(t)) \right)_{1 \leq i,j \leq n}$$

which we treat as a matrix in $M_{n,n}(\mathbf{C}[\![t]\!])$. Our hypothesis implies that $\mathrm{ord}_t \, |Df(\gamma)| = e$.

The next claim is that it is enough to assume that $Df(\gamma)$ is diagonal, of the form

$$Df(\gamma) = \mathrm{Diag}(t^{e_1}, \ldots, t^{e_n}), \quad \sum_{i=1}^{n} e_i = e.$$

Let's assume for now that this is the case, and finish the proof. First, in this case having $Df(\gamma) \cdot \beta \equiv 0 \mod t^{m+1}$ is equivalent to saying that the i-th component of β looks like

$$\beta_i = t^{m+1-e_i} \cdot P_i,$$

where P_i is an arbitrary polynomial in t of degree at most $e_i - 1$. This immediately gives

$$\{\beta \mid Df(\gamma) \cdot \beta \equiv 0 \mod t^{m+1}\} \simeq \mathbf{A}^e = \mathbf{A}^{e_1 + \ldots + e_n}.$$

On the other hand, since $m \geq 2e$, the quadratic terms in (3) all have degree in t higher than $m+1$. Combined with the above, this gives

$$\{\beta \mid f_m(\gamma + \beta) = f_m(\gamma)\} \simeq \mathbf{A}^e.$$

It is also clear that the set $\{\gamma + \beta\}$ is contained in a fiber of π^m_{m-e}, as the description of β_i above implies that $\gamma + \beta \equiv \gamma \mod t^{m-e+1}$.

It remains to see that it is enough to assume that $Df(\gamma)$ is diagonal. As a matrix over $\mathbf{C}[\![t]\!]$, it can be diagonalized; choose $P, Q \in \mathrm{GL}_n\big(\mathbf{C}[\![t]\!]\big)$ such that

$$P \cdot Df(\gamma) \cdot Q = \mathrm{Diag}\ (t^{e_1}, \ldots, t^{e_n}).$$

Write

$$P_m, Q_m \in \mathrm{GL}_n\big(\mathbf{C}[t]/(t^{m+1})\big)$$

for the truncations modulo t^{m+1}. Note that we still have

$$P_m \cdot Df(\gamma) \cdot Q_m = \mathrm{Diag}\ (t^{e_1}, \ldots, t^{e_n}),$$

while multiplying by P_m and Q_m produces automorphisms of $(\mathbf{A}^n)_m$. In particular we have the identification

$$\{\beta \in (\mathbf{A}^n)_m \mid f(\gamma + \beta) \equiv f(\gamma) \mod t^{m+1}\} \simeq$$

$$\simeq \{\delta \in (\mathbf{A}^n)_m \mid P_m \cdot f(\gamma + Q_m \cdot \delta) \equiv P_m \cdot f(\gamma) \mod t^{m+1}\}$$

But now note that

$$P_m \cdot f(\gamma + Q_m \cdot \beta) = P_m \cdot f(\gamma) + P_m \cdot Df(\gamma) \cdot Q_m \cdot \beta + \ldots,$$

which shows that we can reduce to the diagonal setting.

The general case. I will now give the proof in the general case, following [Bl] Appendix A and [Mu] §2.3, which in turn follow [Lo]. For the entire proof, the convenient language is that of derivations, for which a very brief reminder is provided in the Appendix below.

Let $\gamma \in C^e$, with $\gamma_m = \psi_m(\gamma) \in X_m$ and $\psi_0(\gamma) = x \in X$. We first record a crucial interpretation of this information. Note that the birational morphism $f : X \to Y$ induces a short exact sequence

$$0 \longrightarrow f^*\Omega^1_Y \longrightarrow \Omega^1_X \longrightarrow \Omega^1_{X/Y} \longrightarrow 0.$$

The pull-back of this sequence to Spec $\mathbf{C}[\![t]\!]$ via the arc γ is equivalent to the exact sequence

$$(4)\qquad 0 \longrightarrow \Omega^1_{Y,f(x)} \otimes_{\mathcal{O}_{Y,f(x)}} \mathbf{C}[\![t]\!] \xrightarrow{\gamma^* df} \Omega^1_{X,x} \otimes_{\mathcal{O}_{X,x}} \mathbf{C}[\![t]\!] \longrightarrow \Omega^1_{X/Y,x} \otimes_{\mathcal{O}_{X,x}} \mathbf{C}[\![t]\!] \longrightarrow 0.$$

Now by the definition of the contact order, we have that $\det(\gamma^* df) = (t^e)$. Since $\mathbf{C}[\![t]\!]$ is a PID, we can choose bases of the rank n free modules $\Omega^1_{Y,f(x)} \otimes_{\mathcal{O}_{Y,f(x)}} \mathbf{C}[\![t]\!]$ and $\Omega^1_{X,x} \otimes_{\mathcal{O}_{Y,f(x)}} \mathbf{C}[\![t]\!]$ such that $\gamma^*(df)$ is given by a diagonal matrix, and in fact the sequence above becomes

$$(5)\qquad 0 \longrightarrow (\mathbf{C}[\![t]\!])^n \xrightarrow{\mathrm{Diag}(t^{e_1},\ldots,t^{e_n})} (\mathbf{C}[\![t]\!])^n \longrightarrow \bigoplus_{i=1}^n \mathbf{C}[\![t]\!]/(t^{e_i}) \longrightarrow 0$$

with $e_1 + \ldots + e_n = e$.

To prove the statement, let's first assume that we know that $f_m^{-1}(f_m(\gamma_m))$ is contained in a fiber of π^m_{m-e}, and deduce that it is isomorphic to $\mathbf{A}^e$.

Step 1. The first claim is that we have an identification

$$(\pi^m_{m-e})^{-1}(\pi^m_{m-e}(\gamma_m)) \simeq \mathrm{Der}_{\mathbf{C}}\left(\mathcal{O}_{X,x}, (t^{m-e+1})/(t^{m+1})\right).$$

Indeed, given any γ'_m with $\pi^m_{m-e}(\gamma_m) = \pi^m_{m-e}(\gamma'_m)$, we have that the two jets correspond to morphisms $\mathcal{O}_{X,x} \to \mathbf{C}[t]/(t^{m+1})$ such that the compositions with the natural projection

$$\mathcal{O}_{X,x} \longrightarrow \mathbf{C}[t]/(t^{m+1}) \longrightarrow \mathbf{C}[t]/(t^{m-e+1})$$

are equal. This implies that their difference defines a map to the kernel of the projection,

$$\gamma_m - \gamma'_m : \mathcal{O}_{X,x} \longrightarrow (t^{m-e+1})/(t^{m+1})$$

which is easily seen to be a derivation (note that since $m \geq 2e$ the square of the ideal $(t^{m-e+1})/(t^{m+1})$ is zero, which makes the Leibniz rule work).

Step 2. We now show that we have an identification

$$f_m^{-1}(f_m(\gamma_m)) \simeq \mathrm{Der}_{\mathcal{O}_{Y,f(x)}}\left(\mathcal{O}_{X,x}, (t^{m-e+1})/(t^{m+1})\right).$$

Using the sequence (8) below, we have that $\mathrm{Der}_{\mathcal{O}_{Y,f(x)}}(\mathcal{O}_{X,x}, (t^{m-e+1})/(t^{m+1}))$ is the kernel of the natural morphism

$$\mathrm{Der}_{\mathbf{C}}\left(\mathcal{O}_{X,x}, (t^{m-e+1})/(t^{m+1})\right) \longrightarrow \mathrm{Der}_{\mathbf{C}}\left(\mathcal{O}_{Y,f(x)}, (t^{m-e+1})/(t^{m+1})\right),$$

and therefore using Step 1. and the current assumption that

$$f_m^{-1}(f_m(\gamma_m)) \subset (\pi^m_{m-e})^{-1}(\pi^m_{m-e}(\gamma_m)),$$

the identification will be made inside the space $\mathrm{Der}_{\mathbf{C}}(\mathcal{O}_{X,x}, (t^{m-e+1})/(t^{m+1}))$. Consider $\gamma'_m \in (\pi^m_{m-e})^{-1}(\pi^m_{m-e}(\gamma_m))$, so that by Step 1.

$$\gamma'_m - \gamma_m \in \mathrm{Der}_{\mathbf{C}}\left(\mathcal{O}_{X,x}, (t^{m-e+1})/(t^{m+1})\right).$$

Then clearly $\gamma'_m - \gamma_m$ is mapped to 0 in $\mathrm{Der}_{\mathbf{C}}\left(\mathcal{O}_{Y,f(x)}, (t^{m-e+1})/(t^{m+1})\right)$ if and only if $f \circ (\gamma'_m - \gamma_m) = 0$, i.e. if and only if $f_m(\gamma'_m) = f_m(\gamma_m)$.

Step 3. Finally, let's note that

$$\mathrm{Der}_{\mathcal{O}_{Y,f(x)}}\left(\mathcal{O}_{X,x},(t^{m-e+1})/(t^{m+1})\right)\simeq \mathbf{A}^e$$

as claimed. We identify again the space on the left with the kernel of the natural morphism

$$\mathrm{Der}_{\mathbf{C}}\left(\mathcal{O}_{X,x},(t^{m-e+1})/(t^{m+1})\right)\longrightarrow \mathrm{Der}_{\mathbf{C}}\left(\mathcal{O}_{Y,f(x)},(t^{m-e+1})/(t^{m+1})\right),$$

or equivalently, according to the isomorphism in (7), that of the morphism

$$\mathrm{Hom}_{\mathcal{O}_{X,x}}\left(\Omega^1_{X,x},(t^{m-e+1})/(t^{m+1})\right)\longrightarrow \mathrm{Hom}_{\mathcal{O}_{Y,f(x)}}\left(\Omega^1_{Y,f(x)},(t^{m-e+1})/(t^{m+1})\right).$$

This is the same as the natural morphism

$$\mathrm{Hom}_{\mathcal{O}_{X,x}}\left(\Omega^1_{X,x},\mathbf{C}[\![t]\!]/(t^{m+1})\right)\longrightarrow \mathrm{Hom}_{\mathcal{O}_{Y,f(x)}}\left(\Omega^1_{Y,f(x)},\mathbf{C}[\![t]\!]/(t^{m+1})\right)$$

since the jet γ_m has contact of order e with the Jacobian. (As before, the structure of $\mathcal{O}_{X,x}$-module on $\mathbf{C}[\![t]\!]/(t^{m+1})$ is that induced by the jet $\gamma_m:\mathcal{O}_{X,x}\to\mathbf{C}[\![t]\!]/(t^{m+1})$.) Given the sequences (4) and (5), the kernel of this last morphism is isomorphic to

$$\mathrm{Hom}_{\mathcal{O}_{X,x}}\left(\Omega^1_{X/Y,x},\mathbf{C}[\![t]\!]/(t^{m+1})\right)\simeq \mathbf{A}^e.$$

The last isomorphism as a $\mathbf{C}$-space follows via the $\mathbf{C}[\![t]\!]/(t^{m+1})$-module structure, since $m>e$ and we have seen that $\Omega^1_{X/Y,x}\otimes_{\mathcal{O}_{X,x}}\mathbf{C}[\![t]\!]$ is torsion of length e.

We are now left with proving the statement that $f_m^{-1}(f_m(\gamma_m))$ is contained in a fiber of π^m_{m-e}. We again do this in several steps.

Step 4. Let $\gamma\in X_\infty$ such that $\psi_m(\gamma)=\gamma_m$. Consider now $\gamma'_m\in f_m^{-1}(f_m(\gamma_m))$ and $\gamma'\in X_\infty$ such that $\psi_m(\gamma')=\gamma'_m$. To prove what we want, it is enough to construct an element $\tilde{\gamma}\in X_\infty$ such that

$$f_\infty(\tilde{\gamma})=f_\infty(\gamma')\quad\text{and}\quad \psi_{m-e}(\tilde{\gamma})=\psi_{m-e}(\gamma). \tag{6}$$

(Note that this is sufficient since the stability of C^e puts $\tilde{\gamma}$ in C^e, on which f_∞ is injective, and therefore $\tilde{\gamma}=\gamma'$.) To this end, starting with $\gamma^m:=\gamma_m$, we construct by induction on $k\geq m$ a sequence of jets $\gamma^k\in X_k$ satisfying the properties:

(i) $f_k(\gamma^k)=f_k(\psi_k(\gamma'))$.

(ii) $\pi^{k+1}_{k-e}(\gamma^{k+1})=\pi^k_{k-e}(\gamma^k)$.

By the definition of a projective limit, this sequence of jets produces an element $\tilde{\gamma}\in X_\infty$, which then clearly satisfies the conditions (6).

Step 5. Let's first see what are the options for satisfying condition (ii) above. Assume that for some $k\geq m$ we have constructed γ^k. Fix an arbitrary lifting $\beta^{k+1}\in X_{k+1}$ of γ^k. We prove that there is a bijection

$$\{\alpha^{k+1}\in X_{k+1}\mid \pi^{k+1}_{k-e}(\alpha^{k+1})=\pi^k_{k-e}(\gamma^k)\}\longrightarrow \mathrm{Der}_{\mathbf{C}}\left(\mathcal{O}_{X,x},(t^{k-e+1})/(t^{k+2})\right),$$

where we consider $\mathbf{C}[\![t]\!]/(t^{k+2})$ as an $\mathcal{O}_{X,x}$-module via the jet β^{k+1}. Indeed, we can interpret the differences $\alpha^{k+1}-\beta^{k+1}$ as derivations, and the proof of the statement is completely identical to that in Step 1.

Step 6. We now focus on condition (i). We use as before $f_{k+1}(\beta_{k+1})$ as a fixed "anchor" lifting of $f_k(\gamma^k)$. Having done this, as in the previous step all other elements in Y_{k+1} having the same truncation as $f_k(\gamma^k)$ in Y_{k-e} are parametrized by

$$\mathrm{Der}_{\mathbf{C}}\left(\mathcal{O}_{Y,f(x)},(t^{k-e+1})/(t^{k+2})\right).$$

In particular, the other lifting $f_{k+1}(\psi_{k+1}(\gamma'))$ of $f_k(\gamma^k)$ corresponds to such a derivation D. To conclude the inductive step, we need to find an element α^{k+1} as in Step 5. such that $f_{k+1}(\alpha^{k+1})=f_{k+1}(\psi_{k+1}(\gamma'))$ (then we set this element to be γ^{k+1}). We conclude that this is equivalent to showing that D is in the image of the natural map

$$u:\mathrm{Der}_{\mathbf{C}}\left(\mathcal{O}_{X,x},(t^{k-e+1})/(t^{k+2})\right)\longrightarrow \mathrm{Der}_{\mathbf{C}}\left(\mathcal{O}_{Y,f(x)},(t^{k-e+1})/(t^{k+2})\right).$$

Step 7. The natural surjection

$$t^{k-e+1}/t^{k+2}\longrightarrow t^{k-e+1}/t^{k+1}$$

induces a commutative diagram

$$\begin{array}{ccc}\mathrm{Der}_{\mathbf{C}}\left(\mathcal{O}_{X,x},(t^{k-e+1})/(t^{k+2})\right) & \xrightarrow{u} & \mathrm{Der}_{\mathbf{C}}\left(\mathcal{O}_{Y,f(x)},(t^{k-e+1})/(t^{k+2})\right)\\ \downarrow & & \downarrow\\ \mathrm{Der}_{\mathbf{C}}\left(\mathcal{O}_{X,x},(t^{k-e+1})/(t^{k+1})\right) & \xrightarrow{v} & \mathrm{Der}_{\mathbf{C}}\left(\mathcal{O}_{Y,f(x)},(t^{k-e+1})/(t^{k+1})\right)\end{array}$$

To show that $D\in\mathrm{Im}(u)$, let's first observe that $D\in\mathrm{Im}(v)$. Indeed, this follows from the inductive step: the image of D in $\mathrm{Der}_{\mathbf{C}}\left(\mathcal{O}_{Y,f(x)},(t^{k-e+1})/(t^{k+1})\right)$ corresponds to the element $f_k(\psi_k(\gamma'))$, which is the same as $f_k(\gamma^k)$ and hence puts D in the image of v.

Now all that's left to note is that u and v have the same cokernel, so that D also lies in the image of u as desired. But as in the discussion before Step 1. (as well as that in Step 3.), we can identify v, for instance, with the linear transformation of $\mathbf{C}[\![t]\!]/(t^{k+1})$-vector spaces

$$\left(\mathbf{C}[\![t]\!]/(t^{k+1})\right)^n\overset{\mathrm{Diag}(t^{e_1},\ldots,t^{e_n})}{\longrightarrow}\left(\mathbf{C}[\![t]\!]/(t^{k+1})\right)^n,$$

so that

$$\mathrm{Coker}(v)\simeq\bigoplus_{i=1}^n\mathbf{C}[\![t]\!]/(t^{e_i}).$$

The exact same thing can be said about $\mathrm{Coker}(u)$, replacing $k+1$ with $k+2$.

Appendix: Derivations (see e.g. [Ma] §25). Let k be a ring, A a k-algebra, and M an A-module. A *k-derivation* from A to M is a map $D:A\to M$, satisfying the following two properties:

- $D(\lambda(a+b))=\lambda D(a)+\lambda D(b)$ for all $\lambda\in k$ and $a,b\in B$.
- $D(ab)=aD(b)+bD(a)$ for all $a,b\in A$.

The set of all such derivations is denoted $\mathrm{Der}_{\mathbf{C}}(A,M)$. A fundamental property is the isomorphism

$$\mathrm{Der}_{\mathbf{C}}(A,M)\simeq\mathrm{Hom}_A(\Omega_{A/k},M), \tag{7}$$

where $\Omega_{A/k}$ is the A-module of differentials of A over k.

Given an algebra homomorphism $A \to B$ (which in particular turns B into a k-algebra as well) and a B-module N, there is an induced exact sequence

$$0 \longrightarrow \mathrm{Der}_A(B,N) \longrightarrow \mathrm{Der}_{\mathbf{C}}(B,N) \longrightarrow \mathrm{Der}_{\mathbf{C}}(A,N) \tag{8}$$

which is induced via the isomorphism above from the standard exact sequence on differentials

$$\Omega_{A/k} \otimes_A B \longrightarrow \Omega_{B/k} \longrightarrow \Omega_{A/k} \longrightarrow 0.$$

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