MATH 221: PROBLEM SET 9

Due Wednesday, November 18

(1) Let $f\colon R \to S$ be a ring homomorphism.
   (a) If M is a flat S-module and S is flat over R, then M is a flat R-module (by restriction of scalars).
   (b) If M is a flat R-module, then $S \otimes_R M$ is a flat S-module.

(2) If S is a multiplicative system in a ring R, show that $S^{-1}R$ is a flat R-module.

(3) Show that $\mathrm{Ext}^n_{\mathbb{Z}}(A, B) = 0$ for all abelian groups A and B and every $n \geq 2$.

(4) Show that $\mathrm{Tor}^{\mathbb{Z}}_n(A, B) = 0$ for all abelian groups A and B and every $n \geq 2$.

(5) An example of an *infinite* free resolution: show that the following is a free resolution of the module $M = \mathbb{Z}/2\mathbb{Z}$ over $R = \mathbb{Z}/4\mathbb{Z}$ (with the standard module structure):
$$\cdots \xrightarrow{f} \mathbb{Z}/4\mathbb{Z} \xrightarrow{f} \mathbb{Z}/4\mathbb{Z} \xrightarrow{f} \mathbb{Z}/4\mathbb{Z} \xrightarrow{g} \mathbb{Z}/2\mathbb{Z} \to 0,$$
where $f(x) = 2x \pmod 4$ and $g(x) = x \pmod 2$.

(6) By analogy with the case of projective dimension, prove the characterization of injective dimension via the vanishing of Ext, i.e. that for any R-module N and any $n \geq 0$, the following are equivalent:
   (a) $\mathrm{id}_R N \leq n$.
   (b) $\mathrm{Ext}^i_R(M, N) = 0$, for every $i > n$ and every R-module M.
   (c) $\mathrm{Ext}^{n+1}_R(M, N) = 0$, for every R-module M.
   (d) If
$$0 \to N \to E_0 \to E_1 \to \cdots \to E_{n-1} \to Q_n \to 0$$
   is an exact sequence with all E_i injective, then Q_n is injective.

(7) **Extra credit:** Show that an R-module M is flat $\iff$ for any ideal $I \subseteq R$ we have $\mathrm{Tor}^R_1(R/I, M) = 0 \iff$ the natural morphism $I \otimes_R M \to IM$ is injective. (The only point that is challenging is the implication from right to left in the first equivalence.)

(8) Use the previous problem to show that over a PID a module is flat if and only if it is torsion-free.