MATH 137: PROBLEM SET 3

Due Friday, February 17

(1) [Ha] Ch.I: 1.9
(2) Let \( U \) be a nonempty open subset in the affine variety \( X \). Then

\[ \dim U = \dim X. \]

(3) Let \( X \subset \mathbb{A}^m \) and \( Y \subset \mathbb{A}^n \) be affine algebraic sets. Show that \( X \times Y \subset \mathbb{A}^{m+n} \) is an algebraic set, and if \( X \) and \( Y \) are irreducible then \( X \times Y \) is so.

**Extra credit:** Show that the affine coordinate ring of the product satisfies

\[ A(X \times Y) \cong A(X) \otimes_k A(Y), \]

and show the formula \( \dim X \times Y = \dim X + \dim Y \).

(4) (i) Let \( X \subset \mathbb{A}^n \) be an affine variety, and let \( f \in k[X_1, \ldots, X_n] \) such that \( X \not\subset Z(f) \). Show that the *distinguished open set* \( X_f := X \setminus Z(f) \) is isomorphic to an affine subvariety in \( \mathbb{A}^{n+1} \), whose coordinate algebra satisfies

\[ A(X_f) \cong A(X)[X_{n+1}]/(X_{n+1}f - 1). \]

(ii) Show that if we consider all such \( f \), the open sets \( X_f \) form a basis for the topology of \( X \).

(5) Let \( R \subseteq S \subseteq T \) be ring extensions of integral domains. Show that:

(a) If \( S \) is module-finite over \( R \), and \( T \) is module-finite over \( S \), then \( T \) is module finite over \( R \).

(b) If \( S \) is integral over \( R \), and \( T \) is integral over \( S \), then \( T \) is integral over \( R \).

(6) If \( k \) is an algebraically closed field, and \( I \) is an ideal in \( R = k[X_1, \ldots, X_n] \), show that \( Z = Z(I) \subset \mathbb{A}^n \) is a finite set if and only if \( \dim_k R/I < \infty \), i.e. the coordinate algebra \( A(Z) \) is finite dimensional as a \( k \)-vector space.

Note that knowing \( \dim_k R/I \) (usually called the *length of the scheme* defined by \( I \)) is more refined than just knowing \( Z \). For instance, compare \( \dim_k R/I \) with \( R = k[X,Y] \), for

(a) \( I = (X,Y) \) versus \( I = (X^2,Y) \).

(b) \( I = (Y - X, X^2 + Y^2 - 1) \) versus \( I = (Y - X^2, Y^2 - X^3) \).

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\(^1\)This is the same as the localization \( A(X)_f = A(X)[1/f] \) of \( A(X) \) at \( f \); note that we continue to use the notation \( f \) for its image in \( A(X) \), for simplicity.