AN ALGEBRAIC LIFTING INVARIANT OF ELLENBERG, VENKATESH, AND WESTERLAND

MELANIE MATCHETT WOOD

Abstract. We define and prove basic properties of a lifting invariant of curves over an algebraically closed field k with a map to the projective line $\mathbb{P}^1_k$ that was introduced by Ellenberg, Venkatesh, and Westerland.

1. Introduction

In this note, we explain a lifting invariant of curves over an algebraically closed field k with a map to the projective line $\mathbb{P}^1_k$ that was introduced by Ellenberg, Venkatesh, and Westerland, building on ideas of Serre [Ser90] and Fried [Fri95]. A very similar invariant has been defined by Catanese, Lönne, and Perroni [CLP15] for G-covers of curves over the complex numbers. We give a construction in group theory necessary to define the invariant, relate the constructed group to the set of components of certain topological Hurwitz spaces over $\mathbb{C}$, determine the action of automorphisms of k on the invariant, and prove the invariant is constant in families. These results are all based on ideas from the paper [EVW12]. Since the paper [EVW12] was retracted (for issues unrelated to the constructions in the current paper, see the forthcoming Bourbaki seminar by Randal-Williams) we present this note because we think the results are important to have in the literature and we expect them to be of use in future work, especially in using Hurwitz schemes over $\mathbb{F}_q$ to answer questions in the arithmetic statistics of function fields (see, e.g. [BW17, EVW16, ETW17, LT16, Woo17, Woo18] for work along these lines). If G is a finite group and $C \to \mathbb{P}^1_k$ is a tame map of curves with automorphism group G, one can define an invariant of G by the multiset of conjugacy classes of cyclic subgroups of the inertia groups of the map (i.e. monodromy groups around the ramification points). The invariant defined in this paper includes and refines the information of this multiset of conjugacy classes of cyclic subgroups.

In Section 2, we define a first group $U(G, c)$ in which the lifting invariant over $\mathbb{C}$ can be valued. In Section 3, we show how elements of $U(G, c)$ correspond to braid group orbits on tuples of elements of G. This shows how $U(G, c)$ can separate components of a topological Hurwitz space, and generalizes a result of Conway and Parker that appears in a paper of Fried and Vólklein [FV]. In Section 4, we refine the location of where the lifting invariant will lie to a certain twist of $U(G, c)$ by roots of unity. This allows us, in Section 5, to algebraically define the lifting invariant over any algebraically closed field k. In Section 6 we determine how the lifting invariant behaves under change of fields and prove it is constant in families.

1.1. **Notation.** Whenever we require a number should be relatively prime to the characteristic of a field, we mean there to be no condition when the field is characteristic 0.

Notation 1.1. Throughout the paper, let G be a finite group and c set of non-trivial elements of G that is closed under conjugation by G and that generates G. We write D for the set of conjugacy classes in c.

2. Constructing a group where the lifting invariant can be valued

In this section we define a group where the lifting invariant can be valued. Based on [EVW12, Section 7.5], we define a group $U(G,c)$ by presentation with generators $[g]$ for $g \in c$, and relations $[x][y][x]^{-1} = [xyx^{-1}]$ for $x, y \in c$. There is a natural map $U(G,c) \to G$ sending $[g] \to g$, and a natural map $U(G,c) \to \mathbb{Z}^D$ sending $[g]$ to a generator for the conjugacy class of g. There is a natural map $\mathbb{Z}^D \to G^{ab}$ sending a generator for the conjugacy class of $[g]$ to the image of g in the abelianization G^{ab}. Combining, we have a homomorphism

$$U(G,c) \to G \times_{G^{ab}} \mathbb{Z}^D.$$

Lemma 2.1. *Let G, c be as in Notation 1.1. Then $U(G,c) \to G$ is a central extension. Moreover, if $x \in c$ and $y \in G$, with $\tilde{y}$ a preimage of y in $U(G,c)$, then $\tilde{y}[x]\tilde{y}^{-1} = [yxy^{-1}]$.*

Proof. The first claim follows from the second taking $y = 1$, and the fact that c generates G making $U(G,c) \to G$ a surjection. Note that by letting $y = x^{-1}zx$, in the relation we have $[x^{-1}zx] = [x]^{-1}[z][x]$. Let $[g_1]^{a_1} \cdots [g_k]^{a_k}$ be a preimage of y for $g_i \in c$, i.e. $g_1^{a_1} \cdots g_k^{a_k} = y$. Then for $x \in c$, we have, using the defining relation,

$$[g_1]^{a_1} \cdots [g_k]^{a_k}[x][g_k]^{-a_k} \cdots [g_k]^{a_k} = [g_1^{a_1} \cdots g_k^{a_k} x g_k^{-a_k} \cdots g_1^{a_1}] = [yxy^{-1}],$$

which proves the first claim. □

2.1. A more explicit expression for $U(G,c)$.

Next we will see how $U(G,c)$ can be given a more explicit description in terms of Schur covering groups. Central extensions

$$1 \to A \longrightarrow \tilde{G} \longrightarrow G \to 1$$

of G by a finite abelian group A are classified by elements of $H^2(G,A)$. The universal coefficients theorem gives an exact sequence

$$\mathrm{Ext}^1(G^{\mathrm{ab}}, A) \longrightarrow H^2(G,A) \xrightarrow{\pi} \mathrm{Hom}(H_2(G,\mathbb{Z}), A). \tag{1}$$

We write $\tilde{G}$ for the class of the extension in $H^2(G,A)$, with the dependence on the map $\tilde{G} \to G$ implicit.

We recall some standard definitions. A *stem extension* is a central extension $\tilde{G} \to G$, such that the induced map $\tilde{G}^{\mathrm{ab}} \to G^{\mathrm{ab}}$ is an isomorphism. For a finite group G, a *Schur covering group* is a stem extension of H of maximal possible order, or equivalently, a central extension $\tilde{G} \to G$ so that the image $\pi(\tilde{G})$ of the extension class under π above is an *isomorphism* $H_2(G,\mathbb{Z}) \to A$. In general, a Schur cover is not unique.

Given a Schur cover $S \to G$, by definition we have an isomorphism $\pi(S) : H_2(G,\mathbb{Z}) \simeq \ker(S \to G)$. Let $x, y \in G$ be two commuting elements. Then, if $\hat{x}, \hat{y}$ are arbitrary lifts of x, y to S, the commutator $[\hat{x}, \hat{y}]$ lies in $\ker(S \to G)$ and is independent of the choice of $\hat{x}, \hat{y}$. One can check from definitions that in fact $\pi(S)^{-1}([\hat{x}, \hat{y}]) \in H_2(G,\mathbb{Z})$ is independent of the choice of Schur cover because it is the image of the canonical generator for $H_2(\mathbb{Z}^2,\mathbb{Z})$ (i.e. $[(1,0)|(0,1)] - [(0,1)|(1,0)]$ in non-homogeneous chain notation) in the map $H_2(\mathbb{Z}^2,\mathbb{Z}) \to H_2(G,\mathbb{Z})$ induced by the map $\mathbb{Z}^2 \to G$ that sends (i,j) to $x^i y^j$. We denote this element

$$\langle x, y \rangle \in H_2(G,\mathbb{Z}).$$

Definition. Let $H_2(G,c)$ be the quotient of $H_2(G,\mathbb{Z})$ by the subgroup $Q_c \subset H_2(G,\mathbb{Z})$ generated by all elements $\langle x,y\rangle$, where x,y commute and $x \in c$. Given G,c, we define a *reduced Schur cover* $S_c \to G$ to be the quotient of a Schur cover $S \to G$ by $\pi(S)(Q_c)$. For a Schur cover S_c, we have that $\pi(S_c) : H_2(G,\mathbb{Z}) \to \ker(S_c \to G)$ gives an isomorphism $\pi(S_c) : H_2(G,c) \simeq \ker(S_c \to G)$. Like a Schur cover, a reduced Schur cover need not be unique.

Lemma 2.2. *Let G,c be as in Notation 1.1. Let S_c be a reduced Schur cover for G,c. Then the composite map*

$$H_2(S_c,\mathbb{Z}) \to H_2(G,\mathbb{Z}) \to H_2(G,c)$$

is 0*, where the first map is induced from the extension map $S_c \to G$ and the second map is from the definition of $H_2(G,c)$.*

Proof. The map $H^2(G,A) \to H^2(S_c,A)$ induced from $S_c \to G$ takes S_c to the trivial extension $\tilde{S}_c \times_G \tilde{S}_c \to \tilde{G}$, which is split by the diagonal. Since the map (1) is functorial in G, the composite

$$H_2(S_c,\mathbb{Z}) \to H_2(G,\mathbb{Z}) \stackrel{\pi(S_c)}{\to} \ker(S_c \to G)$$

is zero. As remarked above, the map $H_2(G,\mathbb{Z}) \stackrel{\pi(S_c)}{\to} \ker(S_c \to G)$ factors through the quotient $H_2(G,c)$ and gives an isomorphism $H_2(G,c) \simeq \ker(S_c \to G)$. Thus we conclude the lemma. □

Lemma 2.3. *Let G,c be as in Notation 1.1. Let S_c be a reduced Schur cover for G,c. In each conjugacy class in c, pick one element x, and then one preimage $\hat{x}$ of x in S_c. Then if $y = gxg^{-1}$ for $g \in G$, the element $\hat{y} := \tilde{g}\hat{x}\tilde{g}^{-1}$, does not depend on the choice of g or preimage $\tilde{g}$ of g in S_c.*

Proof. Since the extension is central $\tilde{g}\hat{x}\tilde{g}^{-1}$ does not depend on the choice of preimage of g. If $gxg^{-1} = hxh^{-1}$, then $h = gk$ for some k that commutes with x. Note that $[\hat{x},\tilde{k}]$, for any preimage $\tilde{k}$ in S_c of k, is in the image of $\pi(S_c)(Q_c)$, which is trivial. We can take $\tilde{h} = \tilde{g}\tilde{k}$, and this concludes the proof. □

Given a reduced Schur cover S_c, we define $\hat{G} = S_c \times_{G^{\mathrm{ab}}} \mathbb{Z}^D$. (A priori, this depends on the choice of reduced Schur cover.)

Lemma 2.4. *Let G,c be as in Notation 1.1. Let S_c be a reduced Schur cover for G,c and $\hat{G}$ and defined above. Then $\hat{G}^{\mathrm{ab}} = \mathbb{Z}^D$.*

Proof. Since the elements of c generate G^{ab}, for any $g \in S_c$, we can find $(g,z_g) \in \hat{G}$. Thus for $g,h \in S_c$, we have $[(g,z_g),(h,z_h)] = ([g,h],0)$. After we quotient $\hat{G}$ by these commutators, we obtain $S_c^{\mathrm{ab}} \times_{G^{\mathrm{ab}}} \mathbb{Z}^D$. By the fact that a Schur cover is a stem extension so $S_c^{\mathrm{ab}} = G^{\mathrm{ab}}$, and the fact that $\mathbb{Z}^D \to G^{\mathrm{ab}}$ is a surjection, we prove the lemma. □

Now we will see that $\hat{G}$ is a more explicit version of the group $U(G,c)$, and so in fact does not depend on the choice of Schur cover.

Theorem 2.5. *Let G,c be as in Notation 1.1. Let S_c be a reduced Schur cover for G,c and $\hat{G}$ and defined above. We pick lifts of the elements of c to S_c as in Lemma 2.3. There is an isomorphism $\hat{G} \to U(G,c)$ taking $(\hat{x},e_x)$ to $[x]$ for $x \in c$, where e_x is a generator corresponding to the conjugacy class of x.*

Proof. We have a central extension $K \to U(G,c) \to G$ from Lemma 2.1. We have a homomorphism from $\hat{G}$ to G, and whether this lifts to a homorphism $\hat{G} \to U(G,c)$ is equivalent to whether $U(G,c) \in H^2(G,K)$ pulls back to 0 in $H^2(\hat{G},K)$. Consider the commutative diagram

$$\begin{array}{ccc} H^2(G,K) & \xrightarrow{\pi} & \mathrm{Hom}(H_2(G,\mathbb{Z}),K) \\ \downarrow & & \downarrow \\ H^2(\hat{G},K) & \xrightarrow{\sim} & \mathrm{Hom}(H_2(\hat{G},\mathbb{Z}),K) \end{array}$$

where the vertical maps are induced by the map $\hat{G} \to G$. The bottom row is an isomorphism by (1) and Lemma 2.4.

If $u : G \to U(G,c)$ is any section (map of sets), then for $x \in c$ and $y \in G$ with $[x,y] = 1$, we have $\pi(U(G,c))(\langle x,y\rangle) = u(x)u(y)u(x)^{-1}u(y)^{-1}$. Since $[x,y] = 1$, by Lemma 2.1, we have that $\pi(U(G,c))(\langle x,y\rangle) = 0$. Thus $\pi(U(G,c))$ is in the image of $\mathrm{Hom}(H_2(G,c;\mathbb{Z}),K) \subset \mathrm{Hom}(H_2(G,\mathbb{Z}),K)$. Since the composite

$$H_2(\tilde{S}_c,\mathbb{Z}) \to H_2(G,\mathbb{Z}) \to H_2(G,c;\mathbb{Z})$$

is 0 by Lemma 2.2, and $\hat{G} \to G$ factors through $\tilde{S}_c$, we have that the composite

$$H_2(\hat{G},\mathbb{Z}) \to H_2(G,\mathbb{Z}) \to H_2(G,c;\mathbb{Z})$$

is 0.

This implies that $\pi(U(G,c))$ has image 0 in $\mathrm{Hom}(H_2(\hat{G},\mathbb{Z}),K)$ and thus image 0 in $H^2(\hat{G},K)$ by the commutative diagram above. We then have a homomorphism $\phi : \hat{G} \to U(G,c)$ compatible with their maps to G. Now we write $\phi(\hat{x},e_x) = [x]k_x$ for some $k_x \in K$ for one x in each conjugacy class of c. Since by Lemma 2.4 we have that $(\hat{x},e_x)$ for one x from each conjugacy class of c are a free generating set for $\hat{G}^{\mathrm{ab}}$, we have a homomorphism $\psi : \hat{G} \to K$ taking $(\hat{x},e_x) \mapsto k_x$. So $\phi \cdot \psi$ is a homomorphism $\hat{G} \to U(G,c)$ taking $(\hat{x},e_x) \mapsto [x]$ for one x from each conjugacy class of c. Note that if $y = gxg^{-1}$, then $(\hat{y},e_y) = (\tilde{g},z_g)(\hat{x},e_x)(\tilde{g},z_g)^{-1}$ for any lift $\tilde{g}$ to S_c and compatible element z_g (see Lemma 2.3). Since $\phi \cdot \psi(\tilde{g},z_g)$ is a preimage of g in $U(G,c)$, we have $\phi \cdot \psi(\hat{y},e_y) = [gxg^{-1}] = [y]$ by Lemma 2.1.

So we have a homorphism $\hat{G} \to U(G,c)$ taking $(\hat{x},e_x) \mapsto [x]$ for all $x \in c$. We also have a homorphism $U(G,c) \to \hat{G}$ taking $[x] \mapsto (\hat{x},e_x)$ for all $x \in c$, since

$$(\hat{y},e_y)(\hat{x},e_x)(\hat{y},e_y)^{-1} = (\widehat{yxy^{-1}},e_x)$$

for $x,y \in c$ (see Lemma 2.3). So $\phi \cdot \psi : \hat{G} \to U(G,c)$ is a central split extension, and the theorem follows from the fact that $\hat{G}^{ab} \simeq \mathbb{Z}^D \simeq U(G,c)^{ab}$. □

3. Components of Hurwitz spaces and $U(G,c)$

Let G,c be as in Notation 1.1. We define V_n to be the set of all tuples $(g_1,\dots,g_n)$ with $g_i \in c$. The braid group B_n (with generators σ_i for $1 \le i < n$ and relations $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ for $1 \le i \le n-2$ and $\sigma_i\sigma_j = \sigma_j\sigma_i$ for $i-j \ge 2$) acts on V_n where

$$\sigma_i(g_1,\dots,g_n) = (g_1,\dots,g_{i-1},g_ig_{i+1}g_i^{-1},g_i,\dots,g_n).$$

We write $\mathrm{Conf}^n\,\mathbb{C}$ for the topological space of unordered, distinct sets of n elements of $\mathbb{C}$ (obtained as the quotient by the symmetric group S_n of the complement of the big diagonal in $\mathbb{C}^n$). It is well-known that $\pi_1(\operatorname{Conf}^n \mathbb{C}) \simeq B_n$. We define a Hurwitz space $\operatorname{Hur}^n_{G,c}$ to be the covering space of $\operatorname{Conf}^n \mathbb{C}$ whose fiber is V_n and with the action of $\pi_1(\operatorname{Conf}^n \mathbb{C})$ given above. Many authors have studied various quotients of $\operatorname{Hur}^n_{G,c}$ as moduli spaces for branched covers of $\mathbb{P}^1_{\mathbb{C}}$ with certain additional data (see, e.g. [FV, RW06]). In fact $\operatorname{Hur}^n_{G,c}$ is also a moduli space for branched covers of $\mathbb{P}^1_{\mathbb{C}}$ with certain additional data (see [LWZB19]). In any case, by definition, the components of $\operatorname{Hur}^n_{G,c}$ correspond to B_n orbits of V_n.

There is a map of sets $\Pi : V_n \to U(G,c)$ taking $(g_1,\dots,g_n)$ to $[g_1]\cdots[g_n]$, and this map is constant on braid orbits. Then by composition we have a map $V_n/B_n \to \mathbb{Z}^D$. We can also define V_n^G to be the subset of V_n whose coordinates generate G, and V_n^G is preserved set-wise by B_n. The following theorem tells us that at least when there are sufficiently many elements of each conjugacy class, $U(G,c)$ exactly detects braid orbits in V_n^G. In the case when c is all non-trivial elements of G and $H_2(G,c)=1$, the following theorem is a result of Conway and Parker that appeared in the appendix to a paper of Fried and Völklein [FV]. It is remarked in [EVW12] that Fried has an unpublished proof of this result. Also, Kulikov [Kul13] and Lönne [LÏ9] have similar results; yet we are not able to find any complete reference in the literature for the following.

Theorem 3.1. *Let G, c be as in Notation 1.1. Then there is a constant M, such that Π gives a bijection between the elements of V_n^G/B_n and $U(G,c)$ whose coordinates in $\mathbb{Z}^D$ are all at least M.*

Proof. We have that $V := \cup_{n\ge 0} V_n/B_n$ has the structure of a monoid under concatenation, and we see that Π respects products. Since $(g)(g_1,\dots,g_n) = (gg_1g^{-1},\dots,gg_ng^{-1})(g)$ in V, we see that if $h_1\cdots h_n = 1$ in G, then $(h_1,\dots,h_n)$ is central in V. Let $v = \prod_{g\in c}(g)^{\operatorname{ord}(g)}$. Since $(g)^{\operatorname{ord}(g)}$ is central, the order of the product does not matter and v is central.

For $\underline{m} \in \mathbb{Z}^D$, let $V_{\underline{m}}$ be the set of elements of V whose image in $\mathbb{Z}^D$ is $\underline{m}$, and $V^G_{\underline{m}}$ be the subset of $V_{\underline{m}}$ whose coordinates generate G. (Note these are defined as sets of braid orbits.) We write $\min \underline{m}$ for the minimum coordinate of $\underline{m}$, and use max similarly. For $g \in c$, let e_g be the image of g in $\mathbb{Z}^D$. We claim that for $g\in c$ the map $V^G_{\underline{m}} \to V^G_{\underline{m}+e_g}$ that takes $x \mapsto (g)x$ is surjective as long as $\min \underline{m}$ is sufficiently large. There is an M_0 only depending on G such that when $\min \underline{m} \ge M_0$, for $y \in V^G_{\underline{m}+e_g}$ there is some element $h\in G$ in the conjugacy class of g that appears in the coordinates of (any representative of) y at least $\operatorname{ord}(g)+1$ times, and can be braided (with σ_i^{-1}) to the left so that $y = (h)^{\operatorname{ord}(g)}hz$. Then by [FV, Appendix, Lemma 3], since the coordinates of hz generate G, we have that $(h)^{\operatorname{ord}(g)}hz = (g)^{\operatorname{ord}(g)}z'$ for some z'. This proves the claim.

Choose $\underline{f} \in \mathbb{Z}^D$ so that the size of the set of $V^G_{\underline{f}}$ is minimal under the condition that $\min \underline{f} \ge M_0$. Then it follows that for $\underline{m}\in\mathbb{Z}^D$ with $\min \underline{m} \ge \max \underline{f}$ that $V^G_{\underline{f}}$ and $V^G_{\underline{m}}$ are the same size. This is because we can get from the first to the second with a sequence of surjective maps, yet by choice of $\underline{f}$, we have that $V^G_{\underline{f}}$ has size at most the size of $V^G_{\underline{m}}$. Moreover, since surjective maps between sets of the same size are bijective, we have that for every $g\in c$ left concatenation by (g) gives a bijective map $V^G_{\underline{m}} \to V^G_{\underline{m}+e_g}$ for $\underline{m}$ with $\min \underline{m} \ge \max \underline{f}$. It follows that for for $\underline{m}$ with $\min \underline{m} \ge \max \underline{f}$, we have that the map $V^G_{\underline{m}} \to V^G_{\underline{m}+\underline{d}}$ that takes $x \mapsto vx$ is bijective (where $\underline{d}$ is the image of v in $\mathbb{Z}^D$).

From the definition of v, we can see for every $g\in c$ we have $gv_g = v = v_g g$ for some $v_g \in V$. Thus the localization $V[v^{-1}]$ of the monoid V is in fact a group. We have that $\Pi : V[v^{-1}] \to$

$U(G,c)$ is a group homomorphism. Since for $g,h \in c$, we have $(ghg^{-1})(g) = (g)(h)$ in V, the generators (g) of V satisfy the defining relations of $U(G,c)$, giving a homomorphism $U(G,c) \to V[v^{-1}]$ taking $[g]$ to (g), and hence inverting Π and proving it is an isomorphism.

Elements $a, b, \in V$ have equal image in $V[v^{-1}]$ if and only if $v^\ell a = v^\ell b$ in V for some ℓ. Thus we have that for two elements $x, y \in V$ that $\Pi(x) = \Pi(y)$ if and only if $v^\ell x = v^\ell y$ for some ℓ. Thus if $x, y \in V^G_{\underline{m}}$ with $\min \underline{m} \geq \max \underline{f}$, we have that $\Pi(x) = \Pi(y)$ if and only if $x = y$.

Now, we claim there is an M_1 depending only on G such that for $\min \underline{m} \geq M_1$ and $w \in V_{\underline{m}}$, there is a $w' \in V^G_{\underline{m}}$ such that $\Pi(w) = \Pi(w')$. We order the conjugacy classes in c and suppose the first conjugacy class has j elements $h_1, \dots, h_j$. We can choose M_1 such that any $w \in V_{\underline{m}}$ has at least $j \operatorname{ord}(h_1)$ coordinates that are h_i for some i, and then we braid all of those (h_i) to the left, so we can assume $w = (h_i)^{j \operatorname{ord}(h_i)} w_1$. Since $[h_i]$ and $[h_\ell]$ are conjugate in $U(G,c)$ for all i, ℓ, and $[h_i]^{\operatorname{ord}(h_i)}$ is central, we see that $\Pi(w) = \Pi(h_1^{\operatorname{ord}(h_1)} \cdots h_j^{\operatorname{ord}(h_j)} w_1)$. We can proceed similarly for the other conjugacy classes, until we have a $w' \in V$ that contains every element from c among its coordinates and $\Pi(w) = \Pi(w')$.

Finally, suppose we have an element $y \in U(G,c)$ with image $\underline{m} \in \mathbb{Z}^D$, such that $\min \underline{m} \geq \max \underline{f} + M_1$. From $V[v^{-1}] \simeq U(G,c)$, we have that $y = zv^{-\ell}$ where z is in the image of V and ℓ is a non-negative integer. Let z be the image of $w \in V_{\underline{m}+\ell\underline{d}}$. Then choose $w' \in V^G_{\underline{m}+\ell\underline{d}}$ as above so $\Pi(w') = \Pi(w) = z$. Also, we have that $w' = v^\ell x$ for $x \in V^G_{\underline{m}}$. Then we have $\Pi(x) = v^{-\ell}\Pi(w') = y$.

Thus we can conclude that for $\min \underline{m}$ sufficiently large, the map Π gives a bijection between $V^G_{\underline{m}}$ and elements of $U(G,c)$ with image $\underline{m} \in \mathbb{Z}^D$. □

4. More precisely the group where the lifting invariant is valued

In order to define an algebraic lifting invariant that will have an action of the automorphism group of the base field, we must use not $U(G,c)$ but a certain twist of $U(G,c)$. In this section, we will define that twist.

Let k be an algebraically closed field. We define $\hat{\mathbb{Z}}(1)_k = \varprojlim \mu_m(k)$ and $\hat{\mathbb{Z}}_k = \varprojlim \mathbb{Z}/m\mathbb{Z}$, where m ranges over positive integers relatively prime to $\operatorname{char} k$. The subset of topological generators of $\hat{\mathbb{Z}}(1)_k$ will be denoted $\hat{\mathbb{Z}}(1)^\times_k$. It is a torsor for the units $(\hat{\mathbb{Z}}_k)^\times$ of $\hat{\mathbb{Z}}_k$. For a set X with an action of $(\hat{\mathbb{Z}}_k)^\times$, we define

$$X\langle -1 \rangle_k := \operatorname{Mor}_{(\hat{\mathbb{Z}}_k)^\times}(\hat{\mathbb{Z}}(1)^\times_k, X) \tag{2}$$

to be the set of functions $\hat{\mathbb{Z}}(1)^\times_k \to X$ equivariant for the $(\hat{\mathbb{Z}}_k)^\times$ actions. If we choose an element $\mu \in \hat{\mathbb{Z}}(1)^\times_k$, then elements of $X\langle -1 \rangle_k$ are specified by their values on μ.

Let G be a finite group and c a subset of G closed under conjugation by elements of G and closed under invertible powering (e.g. if $g \in c$ and $(m, \operatorname{ord}(g)) = 1$, then $g^m \in c$). We write D for the set of conjugacy classes in c. If $\operatorname{char} k \neq 0$, we require that $\operatorname{char} k \nmid |G|$. Note that $(\hat{\mathbb{Z}}_k)^\times$ acts on the set of elements of G, where $\{\alpha_m\} \in (\hat{\mathbb{Z}}_k)^\times$ takes g to $g^{\alpha_{\operatorname{ord}(g)}}$. There is an induced action of $(\hat{\mathbb{Z}}_k)^\times$ on D and hence $\mathbb{Z}^D$. Similarly, there is an action by powering of $(\hat{\mathbb{Z}}_k)^\times$ on any prime to $\operatorname{char} k$ profinite group. We call all of these actions of $(\hat{\mathbb{Z}}_k)^\times$ the *powering action.*

Let $\hat{U}_k(G,c)$ be the pro-prime-to-(char k) (profinite if char $k=0$) completion of $U(G,c)$. Then we have an action $(\hat{\mathbb{Z}}_k)^\times$ on the group $\hat{U}_k(G,c)$, denoted by $\cdot$ such that

$$\alpha\cdot[g]=[g^{\alpha^{-1}}]^\alpha$$

for $\alpha\in(\hat{\mathbb{Z}}_k)^\times$. To see that this is well-defined, we need to check that the defining relations $[x][y][x]^{-1}=[xyx^{-1}]$ are mapped to relations under this rule. In other words, we need to check that

$$[x^{\alpha^{-1}}]^\alpha[y^{\alpha^{-1}}]^\alpha[x^{\alpha^{-1}}]^{-\alpha}=[x][y^{\alpha^{-1}}]^\alpha[x]^{-1},$$

which holds by Lemma 2.1.

Now we will define an action of $(\hat{\mathbb{Z}}_k)^\times$ on the set of elements of $\hat{U}_k(G,c)$ *(the action is not a group homomorphism)* via the rule

$$\alpha * v=(\alpha^{-1}\cdot v)^\alpha$$

for $v\in(\hat{\mathbb{Z}}_k)^\times$.

We show below that the $*$ action of $(\hat{\mathbb{Z}}_k)^\times$ given above in fact gives an action of $(\hat{\mathbb{Z}}_k)^\times$ on the set $U(G,c)$. From Theorem 2.5, we have an explicit structure of $U(G,c)$ as $S_c\times_{G^{\mathrm{ab}}}\mathbb{Z}^D$, and so every finite index normal subgroup of $U(G,c)$ contains $1\times(m\mathbb{Z})^D$ for some m, and $\hat{U}_k(G,c)=S_c\times_{G^{\mathrm{ab}}}(\hat{\mathbb{Z}}_k)^D$. (Note that any prime not dividing $|G|$ also does not divide $|S_c|$.) So we see that the map from $U(G,c)$ to $\hat{U}_k(G,c)$ is injective and $U(G,c)$ is exactly the subgroup of elements of $\hat{U}_k(G,c)$ whose image in $(\hat{\mathbb{Z}}_k)^D$ lies in $\mathbb{Z}^D$.

The morphism $\hat{U}_k(G,c)\to(\hat{\mathbb{Z}}_k)^D$ is equivariant for $(\hat{\mathbb{Z}}_k)^\times$, where $(\hat{\mathbb{Z}}_k)^\times$ acts on $\hat{U}_k(G,c)$ by means of $*$, and it it acts on $(\hat{\mathbb{Z}}_k)^D$ by the powering action. In other words, the following diagram commutes for any $\alpha\in(\hat{\mathbb{Z}}_k)^\times$:

(3)
$$\begin{array}{ccc}\hat{U}_k(G,c) & \longrightarrow & (\hat{\mathbb{Z}}_k)^D\\ {\scriptstyle\alpha*}\downarrow & & {\scriptstyle\alpha}\downarrow\\ \hat{U}_k(G,c) & \longrightarrow & (\hat{\mathbb{Z}}_k)^D\end{array}$$

This means that the $*$ action preserves $U(G,c)\subset\hat{U}_k(G,c)$. We call this action of $(\hat{\mathbb{Z}}_k)^\times$ on $U(G,c)$ the *discrete action*. When we write $U(G,c)\langle-1\rangle$, it is for the discrete action of $(\hat{\mathbb{Z}}_k)^\times$ on the set $U(G,c)$.

4.1. **Another description of the discrete action.** We have

$$\alpha^{-1}\cdot([g_1]^{\pm_1 1}\cdots[g_m]^{\pm_m 1})=[g_1^\alpha]^{\pm_1\alpha^{-1}}\cdots[g_m^\alpha]^{\pm_m\alpha^{-1}}.$$

Then

$$\alpha*([g_1]^{\pm_1 1}\cdots[g_m]^{\pm_m 1})=\left([g_1^\alpha]^{\pm_1\alpha^{-1}}\cdots[g_m^\alpha]^{\pm_m\alpha^{-1}}\right)^\alpha.$$

Suppose we define $w_i(\alpha)=[g_i]^{-\alpha}[g_i^\alpha]$. Note this is a central element. Then $[g_i^\alpha]=[g_i]^\alpha w_i(\alpha)$. So

$$\begin{aligned}\alpha*([g_1]^{\pm_1 1}\cdots[g_m]^{\pm_m 1})&=\left(([g_1]^\alpha w_1(\alpha))^{\pm_1\alpha^{-1}}\cdots([g_m]^\alpha w_m(\alpha))^{\pm_m\alpha^{-1}}\right)^\alpha\\&=\left([g_1]^{\pm_1 1}\cdots[g_m]^{\pm_m 1}\right)^\alpha\prod_i w_i(\alpha)^{\pm_i 1}.\end{aligned}$$

We can check using Lemma 2.1 that for conjugate $g_i, g_j \in c$, we have $\omega_i(\alpha) = \omega_j(\alpha)$. Thus we have a group homomorphism $\omega_\alpha : \mathbb{Z}^D \to \ker(\hat{U}_k(G,c) \to G)$, sending e_g to $[g]^{-\alpha}[g^\alpha]$, so that for $g \in U(G,c)$ with image $\bar{g} \in \mathbb{Z}^D$, we have

$$\alpha * g = g^\alpha \omega_\alpha(\bar{g}).$$

If we use the isomorphism of Theorem 2.5 to write $U(G,c)$ in the coordinates of $\hat{G} = S_c \times_{G^{ab}} \mathbb{Z}^D$, so $g = (h, \underline{m})$, then we have that

$$\begin{aligned} g^\alpha \omega_\alpha(\bar{g}) &= (h^\alpha, \alpha\underline{m})\omega_\alpha(\underline{m}) \\ &= (h^\alpha, \alpha\underline{m}) \prod_{\gamma \in D} \left([g_\gamma]^{-\alpha}[g_\gamma^\alpha]\right)^{m_\gamma} \\ &= (h^\alpha, \alpha\underline{m}) \prod_{\gamma \in D} \left((\widehat{g_\gamma}, e_\gamma)^{-\alpha}(\widehat{g_\gamma^\alpha}, e_{\gamma^\alpha})\right)^{m_\gamma} \\ &= \left(h^\alpha \prod_{\gamma \in D} \left(\widehat{g_\gamma}^{-\alpha}\widehat{g_\gamma^\alpha}\right)^{m_\gamma}, \underline{m}^\alpha\right), \end{aligned}$$

where $\alpha\underline{m}$ is obtained from $\underline{m}$ by multiplying each coordinate by α, and g_γ is an element from the conjugacy class γ, and e_γ is the standard basis element of $\mathbb{Z}^D$ corresponding to γ. Thus we have that

$$\alpha * (h, \underline{m}) = \left(h^\alpha \prod_{\gamma \in D} \left(\widehat{g_\gamma}^{-\alpha}\widehat{g_\gamma^\alpha}\right)^{m_\gamma}, \underline{m}^\alpha\right). \tag{4}$$

Remark 4.1. From Equation (4), we can see that for $\alpha \in (\hat{\mathbb{Z}}_k)^\times$ such that $\alpha \equiv 1 \pmod{|G|^2}$, we have that α acts trivially on $U(G,c)$ (using that the exponent of the Schur multiplier divides the order of the group). Thus a map in $U(G,c)\langle -1 \rangle$ will have the same image on two different elements of $\hat{\mathbb{Z}}(1)_k^\times$ that have the same image in $\mu_{|G|^2}(k)$.

5. Definition of the lifting invariant

Notation 5.1. Let G be a finite group and c a subset of G closed under conjugation by elements of G and closed under invertible powering (e.g. if $g \in c$ and $(m, \operatorname{ord}(g)) = 1$, then $g^m \in c$). We write D for the set of conjugacy classes in c.

In this section we will define the lifting invariant. First, we need to define precise the objects on which it will be defined.

Let S be a scheme. A *curve* over S is a smooth and proper map $X \to S$ whose geometric fibers are connected and 1-dimensional. A *cover* of a curve X over S is a finite, flat, and surjective morphism $Y \to X$ of S-schemes, where Y is also a curve over S. A cover $f\colon Y \to X$ is *Galois* if f is separable and if $\operatorname{Aut} f$ acts transitively on fibers of geometric points of X. Associated to a cover $Y \to X$ is its *branch locus* $D \subset X$, which has the properties that $D \to S$ is étale, the restriction of f to $X - D$ is étale, and $X - D$ is maximal with respect to this property. If there exists a constant n such that the degree of each geometric fiber of $D \to S$ is equal to n (which is automatic if S is connected), then we say that f has n *branch points*. A cover is *tame* if the ramification index at any point is prime to the characteristic of that point.

A *marked, branched* G cover of $\mathbb{P}^1$ over S is a tame Galois cover X of $\mathbb{P}^1_S$ with n branch points, together with a choice of identification of G with the automorphism group of the cover, and a section $P\colon S \to X$ over the standard infinity section $s_\infty : S \to \mathbb{P}^1_S$, where we also require that $\operatorname{im} s_\infty$ is disjoint from the branch locus of the cover.

5.1. **Inertia groups.** Now we let k be an algebraically closed field. The completion of $k(z)$, for the discrete valuation associated to z, is the field $k((z))$ of Laurent series. The maximal prime-to-$(\operatorname{char} k)$ extension of $k((t))$ is the field $k((z^{1/\infty}))$ of Puiseaux series generated by $z^{1/m}$ for m relatively prime to p. We have that $\operatorname{Gal}(k((z^{1/\infty}))/k((z))) \simeq \hat{\mathbb{Z}}(1)_k$, via $\sigma \mapsto \{\sigma_m\}$, where $\sigma_m = \sigma(z^{1/m})/z^{1/m}$.

Let K be any Galois prime-to-$(\operatorname{char} k)$ extension of $k(t)$. For a $t_0 \in k$, if we let $z = t - t_0$ we have a $\operatorname{Gal}(K/k(t))$-conjugacy class of homorphisms $\operatorname{Gal}(k((z^{1/\infty}))/k((z))) \to \operatorname{Gal}(K/k(t))$ corresponding to the homorphisms $K \to k((z^{1/\infty}))$ respecting $k(t)$, which from the isomorphism above gives a conjugacy class of homomorphisms

$$r_{t_0} : \hat{\mathbb{Z}}(1)_k \to \operatorname{Gal}(K/k(t)) \tag{5}$$

coming from t_0, whose images are the inertia groups of t_0.

5.2. **Generators for π_1.** We continue with k, an algebraically closed field. Let U be an open subset of $\mathbb{P}^1_k$ that includes the point ∞. Denote by $\pi_1'(U, \infty)$ its maximal prime-to-$(\operatorname{char} k)$ quotient of the étale fundamental group of U based at ∞ (i.e. the Galois group of the maximal prime-to-$(\operatorname{char} k)$ extension of $k(t)$ unramified at points of U). Write $t_1, \dots, t_n$ for the k points of $\mathbb{P}^1 - U$, i.e. $t_i \in k$. By Grothendieck's comparison of étale and topological π_1 (see [Gro03, Corollaire 2.12, Exposé XIII]), we have that $\pi_1'(U, \infty)$ contains elements $\gamma_j \in \pi_1'(U, \infty)$ with the property that

$$\gamma_1 \dots \gamma_n = 1 \tag{6}$$

and $\gamma_1, \dots, \gamma_n$ topologically generate inertia groups at $t_1, \dots, t_n$, i.e. are $r_{t_i}(\zeta_i)$ for some $\zeta_i \in \hat{\mathbb{Z}}(1)_k$, and $\pi_1'(U, \infty)$ is free as a prime-to-$(\operatorname{char} k)$ profinite group on generators $\gamma_1, \dots, \gamma_{n-1}$. If we consider the action of the γ_i on the extension

$$k\left(t,\ \sqrt[m]{(t-t_1)/(t-t_2)},\ \sqrt[m]{(t-t_2)/(t-t_3)}, \cdots,\ \sqrt[m]{(t-t_n)/(t-t_1)}\right),$$

we find that $\gamma_1 \dots \gamma_n = 1$ implies that the ζ_i are all equal (and further that these values ζ_i do not depend on the choice of conjugacy class of r_{t_i}). We write $\underline{\gamma}$ for $\gamma_1, \dots, \gamma_n$ and $\mathrm{I}(\underline{\gamma}) \in \hat{\mathbb{Z}}(1)_k$ common value of the ζ_i.

5.3. **Definition of the lifting invariant.** Given a branched, marked G cover X of $\mathbb{P}^1$ over $\operatorname{Spec} k$, let U be the complement of the branch locus in $\mathbb{P}^1_k$, and Y the preimage of U in X. The marked basepoint P of Y, makes $(Y, P) \to (U, \infty)$ a pointed Galois étale map, which gives a surjection $\pi_1'(U, \infty) \to \operatorname{Aut}(Y \to U)$ (where we have a surjection and not just a conjugacy class of surjections because of the choice of P). Note that we have $\operatorname{Aut}(Y \to U) = \operatorname{Aut}(X \to \mathbb{P}^1)$, and thus combining with the identification of the latter with G, we obtain a surjection $\pi_1'(U, \infty) \to G$.

Theorem 5.2. *Let G, c be as in Notation 5.1, and let k be an algebraically closed field of characteristic relatively prime to $|G|$. Let X be a branched, marked G cover of $\mathbb{P}^1$ over* $\operatorname{Spec} k$*, and let U be the complement of the branch locus in $\mathbb{P}^1_k$. Let*

$$\varphi : \pi_1'(U, \infty) \longrightarrow G$$

be the homomorphism associated to the cover. We assume that all inertia groups of the cover are generated by elements of c. Then there is a unique element $\mathfrak{z} \in \ker(U(G,c) \to G)\langle -1 \rangle$, the lifting invariant*, such that for any choice of ordering of the branch points $t_1, \dots, t_n$ and any choice of $\underline{\gamma} = \gamma_1, \dots, \gamma_n \in \pi_1'(U, \infty)$ so that γ_i topologically generates an inertia group at t_i and $\gamma_1 \dots \gamma_n = 1$, we have that $\mathfrak{z}$ sends $I(\underline{\gamma})$ (defined above) to*

$$Z(\underline{\gamma}) := [\varphi(\gamma_1)] \dots [\varphi(\gamma_n)] \in U(G,c).$$

The action of $(\hat{\mathbb{Z}}_k)^\times$ on $\ker(U(G,c) \to G)$ is inherited from the $*$ action of $(\hat{\mathbb{Z}}_k)^\times$ on $U(G,c)$. This is well defined because the map $U(G,c) \to G$ is equivariant for $(\hat{\mathbb{Z}}_k)^\times$ acting with $*$ on $U(G,c)$ and the powering action on G, and thus the $*$ action preserves $\ker(U(G,c) \to G)$.

Proof. Note that, for any choice $\underline{\gamma} = (\gamma_1, \dots, \gamma_n)$, there is a unique morphism $\mathfrak{z}_{\underline{\gamma}} \in U(G,c)\langle -1 \rangle$ that sends $I(\underline{\gamma}) \in (\hat{\mathbb{Z}}_k)^\times$ to $Z(\underline{\gamma}) \in U(G,c)$ (see remark after (2)). The content of Theorem 5.2 is that this morphism is independent of the choices of order of the branch points and $\underline{\gamma}$.

Now consider a different choice $\underline{\gamma}'$ (and implicitly a different choice of ordering of branch points) with $I(\underline{\gamma}') = I(\underline{\gamma})^\alpha$. There is then a permutation $\sigma \in S_n$ such that γ_i' is conjugate to $\gamma_{\sigma(i)}^\alpha$ for all i. Since $\pi_1'(U, \infty)$ is free as a prime-to-$(\operatorname{char} k)$ profinite group, φ lifts to a homomorphism $\tilde{\varphi} : \pi_1'(U, \infty) \to \hat{U}_k(G,c)$. We write $\hat{K}$ for $\ker(\hat{U}_k(G,c) \to G)$, and recall that $\hat{U}_k(G,c) \to G$ is a central extension. We then define

$$z_i' = \tilde{\varphi}(\gamma_i')^{-1}[\varphi(\gamma_i')] \tag{7}$$

and note $z_i' \in \hat{K}$ so are central. So we have

$$Z(\underline{\gamma}') = \tilde{\varphi}(\gamma_1')z_1' \cdots \tilde{\varphi}(\gamma_n')z_n' = \tilde{\varphi}(\gamma_1' \cdots \gamma_n')z_1' \cdots z_n' = z_1' \cdots z_n'. \tag{8}$$

Note that for $g \in \pi_1'(U, \infty)$, we have that $[\varphi(g\gamma_i'g^{-1})] = \tilde{\varphi}(g)[\varphi(\gamma_i')]\tilde{\varphi}(g)^{-1}$ by Lemma 2.1, and thus

$$\tilde{\varphi}(g\gamma_i'g^{-1})^{-1}[\varphi(g\gamma_i'g^{-1})] = \tilde{\varphi}(g)\tilde{\varphi}(\gamma_i')^{-1}[\varphi(\gamma_i')]\tilde{\varphi}(g)^{-1} = \tilde{\varphi}(g)z_i'\phi(g)^{-1} = z_i'.$$

So then since γ_i' is conjugate to $\gamma_{\sigma(i)}^\alpha$, we have $\tilde{\varphi}(\gamma_{\sigma(i)}^\alpha)^{-1}[\varphi(\gamma_{\sigma(i)}^\alpha)] = z_i'$. We then compute

$$\begin{aligned}\alpha^{-1} \cdot Z(\underline{\gamma}) &= \prod [\varphi(\gamma_i)^\alpha]^{\alpha^{-1}} = \prod [\varphi(\gamma_i^\alpha)]^{\alpha^{-1}} = \prod \left(\tilde{\varphi}(\gamma_i^\alpha) z_{\sigma^{-1}(i)}' \right)^{\alpha^{-1}} \\ &= \prod \tilde{\varphi}(\gamma_i) \prod ({z_{\sigma^{-1}(i)}'}^{\alpha^{-1}}) = \left(\prod z_i' \right)^{\alpha^{-1}},\end{aligned}$$

where we use crucially that $\tilde{\varphi}$ is a homomorphism and the z_i' are central. (The products of the non-central terms must be taken in the specified order.) It then follows that $\alpha * Z(\underline{\gamma}) = Z(\underline{\gamma}')$, as desired. □

Remark 5.3. Let $c_{\langle,\rangle}$ be the set of conjugacy classes of cyclic subgroups generated by elements of c. Note that $(\hat{\mathbb{Z}}_k)^\times$ orbits in c all have the same image in the natural map $c \to c_{\langle,\rangle}$. Thus we have $\mathbb{Z}^{c_{\langle,\rangle}}\langle -1 \rangle = \mathbb{Z}^{c_{\langle,\rangle}}$. We have a factorization of the natural maps of sets $c \to D \to$

$c_{\langle,\rangle}$. Thus we have a homomorphism $U(G,c) \to \mathbb{Z}^D \to \mathbb{Z}^{c_{\langle,\rangle}}$, where the discrete action of $(\hat{\mathbb{Z}}_k)^\times$ on $U(G,c)$ covers the trivial action on $\mathbb{Z}^{c_{\langle,\rangle}}$. Thus we have an induced map of sets $U(G,c)\langle -1\rangle \to \mathbb{Z}^{c_{\langle,\rangle}}$. If we apply this in the case $c = G \setminus \{1\}$, then the invariant of Theorem 5.2 has image in $\mathbb{Z}^{c_{\langle,\rangle}}$ which counts the number of times each conjugacy class of cyclic subgroups appears as an inertia group in the cover.

Remark 5.4. Let G, c be as in Notation 5.1, k be any algebraically closed field of characteristic prime to $|G|$, and $t_1, \dots, t_n \in \mathbb{P}^1_k$ distinct closed points. Then we claim there is a topological generator ζ of $\hat{\mathbb{Z}}(1)_k$ such that for every $g_1, \dots, g_n \in c$ with $g_1 \cdots g_n = 1$, there exists a branched, marked G cover of $\mathbb{P}^1$ over $\operatorname{Spec} k$, branched at exactly $t_1, \dots, t_n$, with all inertia groups of the cover generated by elements of c, and with lifting invariant mapping ζ to $[g_1] \cdots [g_n]$. There is a homomorphism $\phi : \pi_1'(\mathbb{P}^1 \setminus \{t_1, \dots, t_n\}, \infty) \to G$ sending $\gamma_i \mapsto g_i$. Taking the unique map of smooth, proper curves over k that corresponds to the extension given by the generic points of the cover given by ϕ gives the desired cover.

Remark 5.5. In [VE10, Section 2.5] and [Woo17, Section 3] in certain cases analogous definitions of lifting invariants are made for all global fields.

6. Properties of the lifting invariant

6.1. **Change of fields and Galois action.** Let G, c, k, X be as in Theorem 5.2 with lifting invariant $\mathfrak{z}$. If $\sigma : k \to K$ is a homomorphism of algebraically closed fields, then the extension of scalars $X_K := X \times_{\operatorname{Spec} k} \operatorname{Spec} K$ has lifting invariant $\mathfrak{z} \circ \sigma^{-1}$, which is a composition

$$\hat{\mathbb{Z}}(1)_K \xrightarrow{\sigma^{-1}} \hat{\mathbb{Z}}(1)_k \xrightarrow{\mathfrak{z}} U(G,c).$$

This follows from the definition of the lifting invariant, the fact that we can choose γ_i compatibly with the map $\pi_1'(U_K, \infty) \to \pi_1'(U, \infty)$ induced by σ, and the fact the inertia group homorphisms defined in (5) are compatible with the maps induced by σ. In particular, if $K = k$, then we would usually write X_σ instead of X_K. Let $\chi(\sigma^{-1})$ be the cyclotomic character of σ^{-1}, i.e. the map $\sigma^{-1} : \hat{\mathbb{Z}}(1)_k \to \hat{\mathbb{Z}}(1)_k$ is powering by $\chi(\sigma^{-1})$. In this case, if X had lifting invariant $\mathfrak{z}$ such that $\mathfrak{z}(\zeta) = g$, for a topological generator $\zeta \in \hat{\mathbb{Z}}(1)_k$, then X_σ has lifting invariant $\mathfrak{z}_\sigma$ such that $\mathfrak{z}_\sigma(\zeta) = \mathfrak{z}(\zeta^{\chi(\sigma^{-1})}) = \chi(\sigma)^{-1} * g$.

6.2. **Invariant constant in families.** Finally, we prove that the lifting invariant is constant in connected families.

Theorem 6.1. *Let G, c be as in Notation 5.1. Let S be a scheme over* $\operatorname{Spec} \mathbb{Z}[|G|^{-1}]$, *and let X be a branched, marked G cover X of $\mathbb{P}^1$ over S, such that at all geometric points of S, the inertia groups of the associated cover are generated by elements of c. Let $\bar{s}_1$ and $\bar{s}_2$ be geometric points of S such that the image of $\bar{s}_2$ is in the closure of the image of $\bar{s}_1$, and let $k(\bar{s}_i)$ be the algebraically closed field of $\bar{s}_i$. Then there is a map of roots of unity*

$$\sigma : \hat{\mathbb{Z}}(1)_{k(\bar{s}_2)} \to \hat{\mathbb{Z}}(1)_{k(\bar{s}_1)}$$

(where by a slight abuse of notation on the right above we take roots of unity in $k(\bar{s}_1)$ but of order relatively prime to the characteristic of $k(\bar{s}_2)$) such that $\mathfrak{z}_{X_{\bar{s}_2}} = \mathfrak{z}_{X_{\bar{s}_1}} \circ \sigma$ (where this is well-defined by Remark 4.1). If S is a k-scheme for some algebraically closed field k, then the $\hat{\mathbb{Z}}(1)_{k(\bar{s}_i)}$ are naturally identified with $\hat{\mathbb{Z}}(1)_k$, and σ respects this identification.

Proof. Let $D \subset \mathbb{P}^1_S$ be the branch locus of $X \to \mathbb{P}^1_S$. By [AGV72, VIII] Corollaire 7.5, there is a map of S-schemes $\bar{S}(\bar{s}_1) \to \bar{S}(\bar{s}_2)$, where $\bar{S}(\bar{s}_i)$ is the strict localization of S at $\bar{s}_i$ (as in [AGV72, XIII, Section 4]). Let $D_{\bar{s}_i} = D \times_S \bar{s}_i$. Since D is étale over S, we have that $D \times_S \bar{S}(\bar{s}_2)$ is étale, and since $\bar{S}(\bar{s}_2)$ is strictly henselian, we have that $D \times_S \bar{S}(\bar{s}_2)$ is isomorphic to a disjoint union of copies of $\bar{S}(\bar{s}_2)$. Using the maps $\bar{s}_1 \to \bar{S}(\bar{s}_1) \to \bar{S}(\bar{s}_2)$ and $\bar{s}_2 \to \bar{S}(\bar{s}_2)$, we have maps $D_{\bar{s}_i} \to D \times_S \bar{S}(\bar{s}_2)$. The latter maps give a bijection between the points of $D_{\bar{s}_i}$ and the components of $D \times_S \bar{S}(\bar{s}_2)$, and hence a bijection between the points of $D_{\bar{s}_1}$ and $D_{\bar{s}_2}$.

There is a specialization morphism

$$\pi_1'(U \otimes S(\bar{s}_1), \infty) \to \pi_1'(U \otimes S(\bar{s}_2), \infty)$$

that takes inertia groups for the points in $D_{\bar{s}_1}$ to inertia groups for the points in $D_{\bar{s}_2}$ according to the bijection above [Gro03, XIII, Lemme 2.11]. Let $\mathcal{O}_{S,\bar{s}_2}$ be the strictly local ring of of S at $\bar{s}_2$, i.e. $\bar{S}(\bar{s}_1) = \operatorname{Spec} \mathcal{O}_{S,\bar{s}_2}$, and let $k(\bar{s}_i)$ be the function field of $\bar{s}_i$. Let $\mu_n = \operatorname{Spec} \mathbb{Z}[x]/(x^n - 1)$. For n relatively prime to the characteristic of $k(\bar{s}_2)$, by a similar argument as above with μ_n in place of D, using the maps $\bar{s}_i \to \bar{S}(\bar{s}_2)$, we obtain a bijection between the points of $\mu_n(\bar{s}_1)$ and $\mu_n(\bar{s}_2)$, and these are compatible, giving the map σ of the theorem. Then it follows from the statements above about inertia groups and the definition of the specialization map that we can choose the γ_i in the definition of the lifting invariant compatibly between $\bar{s}_1$ and $\bar{s}_2$, and hence $\mathfrak{z}_{X_{\bar{s}_2}} = \mathfrak{z}_{X_{\bar{s}_1}} \circ \sigma$. The final statement of the theorem follows from the observation that all of the morphisms involved are S-morphisms, and thus k-morphisms. □

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Department of Mathematics, University of Wisconsin-Madison, 480 Lincoln Drive, Madison, WI 53705 USA

Email address: mmwood@math.wisc.edu