# NOTES ON DUALITIES FOR MASS FORMULAS

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This manuscript had been first intended as Section 3 of the errata to the paper [WY17], but has got eventually separated.

## 1. More examples on dualities

It turned out in the errata that [WY17, Question 5.2] have negative answers. However it might be still interesting to consider a modified question. Consider masses $M(K,\Gamma,f)$ and $M(K,\Gamma,g)$ for two counting systems $f$ and $g$. Suppose that they are admissible as functions of $r$ (see [WY17, Definition 2.1] for the definition of admissible functions). For a real number $d$, let us say that this (ordered) pair of masses satisfy the *$d$-dimensional weak duality* if it satisfies the equality

$$M(K,\Gamma,f)\cdot q^d - M(K,\Gamma,g) = \mathbb{D}(M(K,\Gamma,g))\cdot q^d - \mathbb{D}(M(K,\Gamma,f)).$$

**Question 1.1.** *When does this generalized weak duality hold (even if the strong duality fails)?*

In the above concrete situation of quadratic extensions, the two masses $M(K,\Gamma,\mathbf{v}_{\sigma_n})$ and $M(K,\Gamma,-\mathbf{w}_{\sigma_n})$ satsify the $n$-dimensional weak duality instead of the $2n$-dimensional weak duality. Namely

$$\begin{aligned}(1+q^{-n+1}-q^{-n}+q^{-3n/2+1})q^n-(q+q^{-n/2+1})\\ =(q^{-1}+q^{n/2-1})q^n-(1+q^{n-1}-q^n+q^{3n/2-1}).\end{aligned}$$

Note that, if $\sigma_{n,\mathbb{C}}\colon \Gamma\to \mathrm{GL}_{2n}(\mathbb{C})$ denotes the representation similarly defined say over $\mathbb{C}$, we have

$$\begin{aligned}M(K,\Gamma,\mathbf{v}_{\sigma_n}) = M(K,\Gamma,\mathbf{a}_{\sigma_n}/2) = M(K,\Gamma,\mathbf{a}_{\sigma_{n,\mathbb{C}}}/2) \text{ and}\\ M(K,\Gamma,-\mathbf{w}_{\sigma_n}) = M(K,\Gamma,-\mathbf{w}_{\sigma_{n,\mathbb{C}}}) = M(K,\Gamma,\mathbf{a}_{\sigma_{n,\mathbb{C}}}/2-\mathbf{t}_{\sigma_{n,\mathbb{C}}}),\end{aligned}$$

where $\mathbf{a}$ is the Artin conductor and $\mathbf{t}$ is its tame part. For the left equalities, see [WY15, Corollary 4.9]. The right ones hold because, in the case of permutation representations, the Artin conductor and its tame part are independent of on which field or ring the representation is defined. If $\tau_n\colon\Gamma\to\mathrm{GL}_n(\mathbb{C})$ is the representation given by the diagonal matrix $\mathrm{diag}(-1,\dots,-1)$, then $\sigma_{n,\mathbb{C}}$ is isomorphic to the direct sum of $\tau_n$ and the trivial representation of degree $n$. It follows that $M(K,\Gamma,\mathbf{v}_{\sigma_n}) = M(K,\Gamma,\mathbf{a}_{\tau_n}/2)$ and $M(K,\Gamma,-\mathbf{w}_{\sigma_n}) = M(K,\Gamma,\mathbf{a}_{\tau_n}/2-\mathbf{t}_{\tau_n})$. Thus, for this degree-$n$ representation $\tau_n$, the "strong duality"

$$\mathbb{D}(M(K,\Gamma,\mathbf{a}_{\tau_n}/2-\mathbf{t}_{\tau_n})) = M(K,\Gamma,\mathbf{a}_{\tau_n}/2) \tag{1.1}$$

fails, while the $n$-dimensional weak duality holds, though this representation is no longer defined over $\mathcal{O}_K$.

We show below a few more examples where the strong duality fails but the weak duality holds in some dimension. To show the failure of the strong dualtiy, we need to compute masses explicitly. We use local class field theory for this purpose.

Local class field theory tells us that for a local field $K$ and a cyclic group $C_p$ of prime order $p$, the continuous surjective homomorphisms $\mathrm{Gal}(K^{\mathrm{sep}}/K) \to C_p$ correspond exactly to continuous surjective homomorphisms $K^* \to C_p$. Let $U_0$ be the group of elements of valuation 0 in $K$. For each positive integer $i$, let $U_i$ be the subgroup of $U_0$ of elements of the form $1+x$, where $x$ has valuation at least $i$. Each homomorphism $K^* \to C_p$ has a *conductor* (note this is a different, but related, use of the word "conductor" than in "Artin conductor"), which is the least integer $c \geq 0$ such that $U_c$ is in the kernel of the homomorphism. If the conductor is $c$, then in the corresponding field extension we have the upper ramification groups $G^{-1} = \cdots = G^{c-1} = C_p$ and $G^c = 1$, and the lower ramification groups $G_{-1} = \cdots = G_{c-1} = C_p$ and $G_c = 1$. (All of this can be found in [Ser67].)

Note that both the Artin conductor and the tame part of the Artin conductor are additive in the representation. In particular, for any $n$, we have $M(K, \Gamma, \mathbf{a}_{\tau_n}/2) = M(K, \Gamma, n\mathbf{a}_\tau/2)$ and $M(K, \Gamma, \mathbf{a}_{\tau_n}/2 - \mathbf{t}_{\tau_n}) = M(K, \Gamma, n(\mathbf{a}_\tau/2 - \mathbf{t}_\tau))$ with $\tau := \tau_1$. When the Galois group is $C_p$, then the Artin conductor for a non-trivial irreducible representation is $c$, where $G_{-1} = \cdots = G_{c-1} = C_p$ and $G_c = 1$, and the tame part is 0 if $c = 0$ and 1 if $c \geq 1$. For a trivial representation, both the Artin conductor and the tame part of the Artin conductor are 0.

The next three propositions show examples where $\Gamma$ is the cyclic group of order 2 or 3 and the strong duality fails.

**Proposition 1.2.** *Suppose that $\Gamma = C_2$, the cyclic group of order 2 and that $K$ has characteristic 0 and residue characteristic 2. For a positive integer $n$, consider the representations $\sigma_n \colon \Gamma \to \mathrm{GL}_{2n}(\mathcal{O}_K)$ and $\tau_n \colon \Gamma \to \mathrm{GL}_n(\mathbb{C})$ as above. Then the masses $M(K, \Gamma, \mathbf{v}_{\sigma_n}) = M(K, \Gamma, \mathbf{a}_{\tau_n}/2)$ and $M(K, \Gamma, -\mathbf{w}_{\sigma_n}) = M(K, \Gamma, \mathbf{a}_{\tau_n}/2 - \mathbf{t}_{\tau_n})$ are dual to each other for $n = 2$ but not for $n \neq 2$.*

*Proof.* Note that $K^* \cong \mathbb{Z} \times U_0$, where the map to $\mathbb{Z}$ is given by valuation, and the conductor only depends on the restriction of a homomorphism to $U_0$. The two choices of maps $\mathbb{Z} \to C_2$ will cancel with the $1/2$ fraction in the definition of the mass. Let $e$be the valuation of 2 in $K$. The group $U_0$ has continuous surjections to $C_2$ exactly as follows: $(q-1)q^{m-1}$ surjections of conductor $2m$ for $1 \leq m \leq e$ and $q^e$ characters of conductor $2e+1$ (e.g. see [Ser78, Section 2, Example (a)]). So, for any real number $s$, we compute

$$M(K, \Gamma, s\mathbf{a}_\tau) = 1 + \sum_{m=1}^{e} (q-1)q^{(1-2s)m-1} + q^{e-(2e+1)s}$$

and

$$\begin{aligned} M(K, \Gamma, s(\mathbf{a}_\tau - 2\mathbf{t}_\tau)) &= 1 + \sum_{m=1}^{e} (q-1)q^{(1-2s)m-1+2s} + q^{e-(2e+1)s+2s} \\ &= 1 + \sum_{m=1}^{e} (q-1)q^{(1-2s)(m-1)} + q^{(1-2s)(e-1)-s+1}. \end{aligned}$$

The case $s = n/2$ corresponds to the masses of the propositions. We have

$$\mathbb{D}(M(K, \Gamma, s\mathbf{a}_\tau)) = 1 + \sum_{m=1}^{e} (1-q)q^{(2s-1)m} + q^{-e+(2e+1)s}.$$

Consider the expansions of $\mathbb{D}(M(K, \Gamma, s\mathbf{a}_\tau))$ and $M(K, \Gamma, s(\mathbf{a}_\tau - 2\mathbf{t}_\tau))$ as a finite linear combination of powers of $q$. The former has only positive powers for $s > 1$.

However, when $n/2 = s > 1$, the latter has also a negative power. Indeed, the lowest exponent is $(1-2s)(e-1)-s+1$, which is negative. Therefore $\mathbb{D}(M(K, G\Gamma, s\mathbf{a}_\tau) \neq M(K, \Gamma, s(\mathbf{a}_\tau - 2\mathbf{t}_\tau))$ for $s > 1$. By direct computation, we see that when $s = 1$,

$$\mathbb{D}(M(K, \Gamma, s\mathbf{a}_\tau)) = M(K, \Gamma, s(\mathbf{a}_\tau - 2\mathbf{t}_\tau)) = 1 + q$$

and when $s = 1/2$,

$$\mathbb{D}(M(K, \Gamma, s\mathbf{a}_\tau)) = e + 1 + q^{1/2} - eq,$$
$$M(K, \Gamma, s(\mathbf{a}_\tau - 2\mathbf{t}_\tau)) = 1 - e + q^{1/2} + eq.$$

□

**Proposition 1.3.** *Let $\Gamma = C_3$ and let $\tau\colon \Gamma \to \mathrm{GL}_1(\mathbb{C})$ be a nontrivial 1-dimensional representation. For a positive integer $n$, let $\tau_n := \tau^{\oplus n}$. Suppose that 3 has valuation 1 in $K$. Then $M(K, \Gamma, \mathbf{a}_{\tau_n}/2)$ and $M(K, \Gamma, \mathbf{a}_{\tau_n}/2 - \mathbf{t}_{\tau_n})$ are not dual to each other for any $n$. (Note that it does not affect any of the functions $\mathbf{a}$, $\mathbf{t}$ and these masses, which of the two irreducible representations we choose as $\tau$.)*

*Proof.* To count the continuous homomorphisms $U_0 \to C_3$ of various conductors (which we have reduced the problem of computing the masses to by the discussion above), we need to determine the 3-rank of $U_0/(U_0^3 U_i)$ for each $i$. Since $U_0/U_1$ is isomorphic to the multiplicative group of the residue field (and thus has order relatively prime to 3, there are no maps $U_0 \to C_3$ of conductor 1, and we have reduced the problem to determining the 3-rank of $U_1/(U_1^3 U_i)$ for each $i \geq 2$.

From [FV02, Chapter I, Section (5.7)], we have that $U_1^3 = U_2$. Thus $U_1/(U_1^3 U_i) \cong U_1/(U_1^3) \cong U_1/U_2$ for all $i \geq 2$, and thus all non-trivial homomorphisms $U_0 \to C_3$ have conductor 2. Moreover, $U_1/U_2$ is a group of exponent 3 and size $q$ (Section (5.4) of same book chapter), so has $q-1$ non-trivial homomorphisms to $C_3$. Since $U_1^3 = U_2$, there are no homorphisms of higher conductor.

Thus we have

$$M(K, \Gamma, s\mathbf{a}_\tau) = 1 + \frac{q-1}{q^{2s}}$$

and

$$M(K, \Gamma, s(\mathbf{a}_\tau - 2\mathbf{t}_\tau)) = q.$$

We can see if we take the dual of $M(K, \Gamma, s\mathbf{a}_\tau)$, we obtain

$$1 + \frac{q^{-1} - 1}{q^{-2s}},$$

which is less than 1 for $q > 1$. □

**Proposition 1.4.** *Suppose that 3 has valuation 2 in $K$ and that $K$ has cube roots of unity, and $\Gamma = C_3$. Let $\tau_n$ be as in the last proposition. Then $M(K, \Gamma, \mathbf{a}_{\tau_n}/2)$ and $M(K, \Gamma, \mathbf{a}_{\tau_n}/2 - \mathbf{t}_{\tau_n})$ are dual to each other for $n = 2$ but not for $n \neq 2$.*

*Proof.* We proceed as in the above proof. We have no homomorphisms $U_0 \to C_3$ of conductor 1, and we need to determine the 3-rank of $U_1/(U_1^3 U_i)$ for each $i \geq 2$ to determine the number of homomorphisms of each higher conductor. From [FV02, Chapter I, Section (5.7)], we have that $U_1^3 \subset U_3$ and $U_2^3 = U_4$. So, $U_1/(U_1^3 U_2) \cong U_1/U_2$, which is exponent 3 and order $q$. Thus there are $q-1$ homormorphisms $U_0 \to C_3$ of conductor 2. Also, $U_1/(U_1^3 U_3) \cong U_1/U_3$, which is an exponent 3 group (from the first description), and order $q^2$ (from the second description). So there are $q^2 - q$ homormorphisms $U_0 \to C_3$ of conductor 3. Finally, $U_1^3 U_3/U_1^3 U_4 = U_3/U_1^3 U_4$, and the same section of the book referenced above tells us that the map $U_1^3 \to U_3/U_4$ has image of size $q/3$ and $U_3/U_4$ has size $q$. Thus, we conclude $U_1^3U_3/U_1^3U_4$ has size 3, and thus $U_1/(U_1^3U_4)$ has exponent 3 and size $3q^2$. That shows there are $2q^2$ homormorphisms $U_0 \to C_3$ of conductor 4. Since $U_2^3 = U_4$ there are no homormorphisms of higher conductor.

Thus we conclude

$$M(K,\Gamma,s\mathbf{a}_\tau) = 1+\frac{q-1}{q^{2s}}+\frac{q^2-q}{q^{3s}}+\frac{2q^2}{q^{4s}} = 1+q^{1-2s}-q^{-2s}+q^{2-3s}-q^{1-3s}+2q^{2-4s}$$

and

$$M(K,\Gamma,s(\mathbf{a}_\tau-2\mathbf{t}_\tau)) = q+\frac{q^2-q}{q^s}+\frac{2q^2}{q^{2s}} = q+q^{2-s}-q^{1-s}+2q^{2-2s}.$$

Then

$$\mathbb{D}(M(K,\Gamma,s\mathbf{a}_\tau)) = 1+q^{2s-1}-q^{2s}+q^{3s-2}-q^{3s-1}+2q^{4s-2}.$$

When $n = 2s = 2$, we see we have duality. When $n = 2s > 2$, the only term with positive exponent of $q$ in $M(K,\Gamma,s(\mathbf{a}_\tau-2\mathbf{t}_\tau))$ is $q^1$, however in $(M(K,\Gamma,s\mathbf{a}_\tau))$ the coefficients of the terms $q^i$ for $i > 0$ sum to 2, and so we don't have duality. When $n = 2s = 1$, then $M(K,\Gamma,s(\mathbf{a}_\tau-2\mathbf{t}_\tau))$ is a polynomial in $q^{1/2}$, but $\mathbb{D}(M(K,\Gamma,s\mathbf{a}_\tau))$ is not, since it has the term $q^{3s-2} = q^{-1/2}$. □

All the cases in the last three propositions satisfy the weak duality in some dimension. We can prove a little more general result:

**Proposition 1.5.** *Let $K$ be a local field of residue characteristic $p > 0$. Let $\sigma\colon \Gamma \to \mathrm{GL}_d(\mathbb{C})$ be a permutation representation of a finite group $\Gamma$ and $\tau\colon \Gamma \to \mathrm{GL}_e(\mathbb{C})$ a summand of $\sigma$ complementary to the largest trivial subrepresentation of $\sigma$ (that is, the trivial $\Gamma$-action on the fixed locus $(\mathbb{C}^d)^\Gamma$). Suppose that for any element $\gamma \in \Gamma$ of order $p$, the fixed point locus is $(\mathbb{C}^e)^\gamma = \{0\}$. Then the masses $M(K,\Gamma,\mathbf{a}_\tau/2)$ and $M(K,\Gamma,\mathbf{a}_\tau/2-\mathbf{t}_\tau))$ satisfy the $e$-dimensional weak duality, provided that these masses are admissible functions.*

*Proof.* Let us denote these masses by $M$ and $N$ respectively. We decompose $M$ as $M = M_t + M_w$, where $M_t$ (resp. $M_w$) is the part of those maps $\rho\colon G_K \to \Gamma$ such that $\rho(I_K)$ is tame (resp. wild), where $I_K$ is the inertia subgroup of $G_K$. Similarly we write $N = N_t + N_w$. We claim that the dualtiy

$$\mathbb{D}(M_t) = N_t \tag{1.2}$$

holds. Indeed, the involution $\iota$ in [WY17, Section 5, the tame case] preserves the subset of $S_{K,\Gamma}$ contributing $M_t$ and $N_t$ and the above duality follows from Lemma 5.1 of the same paper and the equalities $\mathbf{v}_\sigma = \mathbf{a}_\sigma/2 = \mathbf{a}_\tau/2$ and $-\mathbf{w}_\sigma = \mathbf{a}_\sigma/2-\mathbf{t}_\sigma = \mathbf{a}_\tau/2-\mathbf{t}_\tau$.

From the assumption, for any $\rho\colon G_K \to \Gamma$ with wild inertia image, we have $\mathbf{t}(\rho) = e$. Therefore $N_w = M_w \cdot q^e$, dually $\mathbb{D}(N_w)\cdot q^e = \mathbb{D}(M_w)$. Therefore

$$M_w \cdot q^e - N_w = 0 = \mathbb{D}(N_w)\cdot q^e - \mathbb{D}(M_w). \tag{1.3}$$

This implies

$$M \cdot q^e - N = M_t \cdot q^e - N_t,$$
$$\mathbb{D}(N)\cdot q^e - \mathbb{D}(M) = \mathbb{D}(N_t)\cdot q^e - \mathbb{D}(M_t).$$

From (1.2), the right sides of these two equalities are equal. This proves the proposition. □

## References


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