MATH 273 FALL 2020: HOMEWORK 6

Due Wednesday, Oct 21

(1) Let $V = (\mathbb{Z}/\ell\mathbb{Z})^{2n}$ and $A = (\mathbb{Z}/\ell\mathbb{Z})^k$. Let J be the standard element of $\wedge^2 V$ for some choice of basis, i.e. if the basis is x_i, then $J = \sum_{i=1}^n x_i \wedge x_{i+n}$. For any element $\alpha \in \wedge^2 A$, if $n \geq k$, show that there exists a surjection $f : V \to A$, such that $f(J) = \alpha$. (In class we claimed $f(J)$ gave an orbit invariant of the $Sp(V)$ action on $\mathrm{Sur}(V, A)$, and this shows all possible invariants arise, at least for elementary ℓ-groups.)

(2) Let $\mathbb{G}_n$ be the subgroup of $\mathrm{GL}_n(\mathbb{Z}/\ell\mathbb{Z})$ fixing the first basis vector. Determine the orbits of G_n on $\mathrm{Sur}((\mathbb{Z}/\ell\mathbb{Z})^n, (\mathbb{Z}/\ell\mathbb{Z})^k)$ (where the action is by pre-composition) for $n \geq k + 1$.

(3) If M_n is a uniform random element of G_n, what is $\mathbb{E}(\# \mathrm{Sur}(\mathrm{cok}(1 - M_n), (\mathbb{Z}/\ell\mathbb{Z})^k))$ for $n \geq k + 1$? Give a random matrix from a different distribution that has these same moments in the limit as $n \to \infty$ (i.e. describe another sequence X_n of random matrices where $\lim_{n\to\infty} \mathbb{E}(\# \mathrm{Sur}(\mathrm{cok}\, X_n, (\mathbb{Z}/\ell\mathbb{Z})^k))$ is the same as your answer above, but for which the X_n come from a genuinely different distributions than $1 - M_n$)

(4) Give an example of an infinite upper triangular matrix $M \in \mathbb{R}^{\mathbb{N}\times N}$ with 1's along the diagonal, and a non-trivial solution to $Mx = 0$ for some $B \in \mathbb{R}^{\mathbb{N}}$.