MATH 273 FALL 2020: HOMEWORK 5

Due Wednesday, Oct 14

(1) Let K be a number field and d_n be the number of continuous homomorphisms in $\mathrm{Hom}(\prod_v \mathcal{O}_v^*, \mathbb{Z}/3\mathbb{Z})$ such that n is the product of the norms of the ramified primes (in K). Give an Euler product for $D(s) := \sum_{\substack{n\geq 1 \\ 3\nmid n}} d_n n^{-s}$.

(2) Let K be a real quadratic field and e_n be the number of continuous homomorphisms in $\mathrm{Hom}((\prod_v \mathcal{O}_v^*)/O_K^*, \mathbb{Z}/3\mathbb{Z})$ such that n is the product of the norms of the ramified primes (in K). Express $E(s) := \sum_{\substack{n\geq 1 \\ 3\nmid n}} e_n n^{-s}$ as a finite linear combination of Euler products. (Hint: one can take a fundamental unit in O_K^* and sum over characters on $\mathbb{Z}/3\mathbb{Z}$ to make sure it has trivial image.)

(3) For a non-Galois cubic field K with F the quadratic field in K's Galois closure L, and f the conductor (in the sense of class field theory) of L/F, which is defined to be in K but turns out to be an ideal of $\mathbb{Q}$, prove that $\mathrm{Disc}_K = f^2\,\mathrm{Disc}_F$.

(4) If X is a random elementary abelian p-group with moments $\mathbb{E}(\mathrm{Sur}(X, (\mathbb{Z}/p\mathbb{Z})^k)) = 1$, give a formula for $\mathbb{E}(|X|^m)$. Give an actual formula in terms of p and m, not just something like "the number of subspaces...." (Remark: Previously, researchers were computing moments like these $\mathbb{E}(|X|^m)$ without realization that they could be simplified into the Sur moments.)