MATH 273 FALL 2020: HOMEWORK 4

Due Wednesday, Oct 7

(1) Let $K/\mathbb{F}_q(t)$ be a finite, separable extension. Let $\mathcal{O}_K$ be the integral closure of $\mathbb{F}_q[t]$ in K. We can define the ideal $\mathrm{Disc}(\mathcal{O}_K/\mathbb{F}_q[t])$ of $\mathbb{F}_q[t]$ in the usual way. Let v be a place of $\mathbb{F}_q(t)$ (so comes from a discrete valuation). Let Z_v be the elements of $\mathbb{F}_q(t)$ with non-negative valuation at v, and O_v be the integral closure of Z_v in K. We can define local discriminants $\mathrm{Disc}(\mathcal{O}_v/Z_v)$, which are some power of the maximal ideal of Z_v. What is the relationship between $\mathrm{Disc}(\mathcal{O}_K/\mathbb{F}_q[t])$ and a formal product of $\mathrm{Disc}(\mathcal{O}_v/Z_v)$ over all v?

(2) Let K be as above and also $K/\mathbb{F}_q(t)$ quadratic and not characteristic 2. Let g be the genus of the smooth projective curve corresponding to K, and show $q^{2g+2} = \prod_v \mathrm{Nm}(\mathrm{Disc}(\mathcal{O}_v/Z_v))$. (Here $\mathrm{Nm}(I) := |Z_v/I|$.)

(3) For a finite abelian p-group A, the group $\mathrm{GL}_n(\mathbb{Z}_p)$ acts on $\mathrm{Sur}(\mathbb{Z}_p^n, A)$ by $g(\phi) = \phi \circ g^{-1}$. Show that this action is transitive.

(4) For a finite abelian group A of odd order, how many splittings of the exact sequence

$$1 \to A \to A \rtimes_{-1} \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 1$$

are there? (Hint: Schur-Zassenhaus tells us about these.)

(5) Use $\hat{C}_{\mathbb{Q}} \simeq \prod_p \mathbb{Z}_p^*$ to express the following Dirichlet series $D(s)$ as an Euler product. We let $D(s) = \sum_n d_n n^{-s}$, where d_n is the number of (continuous) homomorphisms $\Phi : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathbb{Z}/3\mathbb{Z}$ such that n is the product of the rational primes ramified in ϕ. (Remark: except for $\Phi = 1$, these homomorphisms correspond exactly to pairs K, ϕ with $K \subset \bar{\mathbb{Q}}$ Galois over $\mathbb{Q}$ and ϕ an isomorphism $\mathrm{Gal}(K/\mathbb{Q}) \to \mathbb{Z}/3\mathbb{Z}$ such that n is the product of the rational primes ramified in K.)

(6) Let χ be a primitive Dirichlet character of modulus 3. Show that $D(s)/(\zeta(s)L(s,\chi))$ is holomorphic in $Re(s) > 1/2$. (Remark: With this knowledge, you could apply a Tauberian theorem to conclude that $\sum_{n \leq X} d_n \sim cX$.)