MATH 273 FALL 2020: HOMEWORK 3

Due Wednesday, Sep 30

(1) Let M_n be a random matrix from Haar measure on $M_{n\times n+u}(\mathbb{Z}_p)$ (i.e. each coordinate of the $n\times(n+u)$ matrix is taken independently from the p-adic probability measure on $\mathbb{Z}_p$). For a finite abelian p-group A, find $\mathbb{E}(\#\operatorname{Sur}(\operatorname{cok} M_n, A))$. (Hint: in class we did this with Sur replaced by Isom.)

(2) For random finite abelian p-group X taken from the probability measure in which $\mathbb{P}(X\simeq A) = |A|^{-u}|\operatorname{Aut} A|^{-1}\prod_{i\geq u+1}(1-p^{-i})$, find $\mathbb{E}(\#\operatorname{Sur}(X, A))$. (Hint: use the above problem, and make sure to use some argument if you wish to interchange a sum and limit. You may use the computation of $\operatorname{Prob}(\operatorname{cok} M_n \simeq A)$ from class modified for general u, even though we did it only explicitly for $u = 1$.)

(3) Let S_n be a random matrix from Haar measure on $\operatorname{Sym}_{n\times n}(\mathbb{Z}_p)$ (symmetric $n\times n$ matrices, i.e. we have a symmetric random matrix for which each i, j coordinate of the matrix with $i \geq j$ is taken independently from the p-adic probability measure on $\mathbb{Z}_p$). For a finite abelian p-group A, find $\mathbb{E}(\#\operatorname{Sur}(\operatorname{cok} S_n, A))$.

(4) Prove these is no proper integrally closed sub-ring of $\mathbb{F}_q[t]$ whose fraction field is $\mathbb{F}_q(t)$.

(5) For a Dedekind domain D, prove $Cl(D) \simeq \operatorname{Pic}(D)$. Here Cl means fractional ideals modulo principal ideals, and Pic means the group of finitely generated locally free rank 1 D-modules with tensor product as the group operation. (A module M is locally free rank 1 if for every prime ideal $\wp$ of D the localization $M_\wp$ is a free rank 1 $D_\wp$-module. Remark: finitely generated plus locally free of rank 1 is equivalent to "scheme theoretically locally free of rank 1," i.e. free of rank 1 on an open cover of $\operatorname{Spec} D$.)

(6) For a prime ideal of $\mathbb{F}_q[t]$, the p-adic valuation on $\mathbb{F}_q[t]$ gives a place of $\mathbb{F}_q(t)$. Similarly, if $f(t)$ is a polynomial (of positive degree) in $\mathbb{F}_q[t]$, the prime ideals of $\mathbb{F}_q[1/f(t)]$ give places of $\mathbb{F}_q(t)$. Which primes of $\mathbb{F}_q[t]$ give places that also come from $\mathbb{F}_q[1/f(t)]$? For these primes, give the map from primes of $\mathbb{F}_q[t]$ to primes of $\mathbb{F}_q[1/f(t)]$ that give the same place. Describe the primes of each of $\mathbb{F}_q[t]$ and of $\mathbb{F}_q[1/f(t)]$ that give places that do not come from primes of the other ring.