MATH 273 FALL 2020: HOMEWORK 2

Due Wednesday, Sep 23

(1) Some basic facts:
   (a) Let $M \in M_{n\times n}(\mathbb{Z})$. Show $\operatorname{cok}(M : \mathbb{Z}^n \to \mathbb{Z}^n) \otimes \mathbb{Z}_p \simeq \operatorname{cok}(M : \mathbb{Z}_p^n \to \mathbb{Z}_p^n)$.
   (b) If A is a finite abelian group, show that $A \otimes \mathbb{Z}_p \simeq A[p^\infty]$ (the later is the subgroup of all p-power order elements).
   (c) If A is an finitely generated but not finite abelian group, how do $A \otimes \mathbb{Z}_p$ and $A[p^\infty]$ compare, and could we "improve" the definition of $A[p^\infty]$ so they would be the same?

(2) Let $\mathcal{A}$ be the set of (isom. classes of) profinite abelian groups whose Sylow-p subgroups (inverse limit of p-group quotients) are finite for each p. Show that the natural map $\mathcal{A} \to \prod_p \{$ fin. ab. p-groups $\}$ taking the Sylow p-subgroups is a bijection.

(3) Now consider the topology on $\mathcal{A}$ (finer that the product topology) where a basis of opens is given as $\prod_p U_p$, for any $U_p \subset \{$ fin. ab. p-groups $\}$ where for sufficiently large p, we have that U_p contains both the trivial group and the group of order p. Show that is this topology the characteristic function of pro-cyclic groups is a continuous function but the characteristic function of finite groups is not continuous.

(4) For every positive integer X, let μ_X be a sequence of probability measures on a countable set $\mathcal{G}$. Suppose that for each $A \in \mathcal{G}$ the limit $\lim_{X\to\infty} \mu_X(A)$ exists, and call it $\mu(A)$. Further, suppose that

$$\sum_{A\in\mathcal{G}} \mu(A) = 1.$$

Prove for any bounded function f,

$$\lim_{X\to\infty} \sum_{A\in\mathcal{G}} f(A)\mu_X(A) = \sum_{A\in\mathcal{G}} f(A)\mu(A).$$

("Once we have no escape of mass, we can exchange the limit and sum for any bounded function.")

(5) Let K, L be quadratic extensions of $\mathbb{Q}$. Choose an embedding $\bar{\mathbb{Q}} \to \bar{\mathbb{Q}}_2$ so we have a map $\operatorname{Gal}(\bar{\mathbb{Q}}_2/\mathbb{Q}_2) \to \operatorname{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$. Let I_2 be the inertia group of $\operatorname{Gal}(\bar{\mathbb{Q}}_2/\mathbb{Q}_2)$, and consider the composite maps $\phi_K : I_2 \to \operatorname{Gal}(K/\mathbb{Q})$ and $\phi_L : I_2 \to \operatorname{Gal}(L/\mathbb{Q})$. Show that KL/K is unramified at 2 if and only if $\ker \phi_K \subset \ker \phi_L$.

(6) Using an argument like the one in class to construct $L/\mathbb{Q}$ with LK/K unramified at finite primes, show that if K is real quadratic and its discriminant has k prime divisors, then the 2-rank of the narrow class group of K is $k - 1$. (You can reference the argument in class and just explain what needs to be changed for this case.)

(7) Give a formula for $\#\operatorname{Sur}(\mathbb{Z}_p^n, A)$ for a finite abelian p-group A (in terms of any convenient numerical invariants of $|A|$).