Due Wednesday, Sep 23

(1) Some basic facts:
   (a) Let $M \in M_{n \times n}(\mathbb{Z})$. Show $\text{cok}(M : \mathbb{Z}^n \to \mathbb{Z}^n) \otimes \mathbb{Z}_p \simeq \text{cok}(M : \mathbb{Z}_p^n \to \mathbb{Z}_p^n)$.
   (b) If $A$ is a finite abelian group, show that $A \otimes \mathbb{Z}_p \simeq A[p^\infty]$ (the later is the subgroup of all $p$-power order elements).
   (c) If $A$ is an finitely generated but not finite abelian group, how do $A \otimes \mathbb{Z}_p$ and $A[p^\infty]$ compare, and could we “improve” the definition of $A[p^\infty]$ so they would be the same?

(2) Let $A$ be the set of (isom. classes of) profinite abelian groups whose Sylow-$p$ subgroups (inverse limit of $p$-group quotients) are finite for each $p$. Show that the natural map $A \to \prod_p \{ \text{fin. ab. } p\text{-groups} \}$ taking the Sylow $p$-subgroups is a bijection.

(3) Now consider the topology on $A$ (finer that the product topology) where a basis of opens is given as $\prod_p U_p$, for any $U_p \subset \{ \text{fin. ab. } p\text{-groups} \}$ where for sufficiently large $p$, we have that $U_p$ contains both the trivial group and the group of order $p$. Show that this topology the characteristic function of pro-cyclic groups is a continuous function but the characteristic function of finite groups is not continuous.

(4) For every positive integer $X$, let $\mu_X$ be a sequence of probability measures on a countable set $\mathcal{G}$. Suppose that for each $A \in \mathcal{G}$ the limit $\lim_{X \to \infty} \mu_X(A)$ exists, and call it $\mu(A)$. Further, suppose that

$$\sum_{A \in \mathcal{G}} \mu(A) = 1.$$ 

Prove for any bounded function $f$,

$$\lim_{X \to \infty} \sum_{A \in \mathcal{G}} f(A) \mu_X(A) = \sum_{A \in \mathcal{G}} f(A) \mu(A).$$

(“Once we have no escape of mass, we can exchange the limit and sum for any bounded function.”)

(5) Let $K, L$ be quadratic extensions of $\mathbb{Q}$. Choose an embedding $\overline{\mathbb{Q}} \to \overline{\mathbb{Q}}_2$ so we have a map $\text{Gal}(\overline{\mathbb{Q}}_2/\mathbb{Q}_2) \to \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Let $I_2$ be the inertia group of $\text{Gal}(\overline{\mathbb{Q}}_2/\mathbb{Q}_2)$, and consider the composite maps $\phi_K : I_2 \to \text{Gal}(K/\mathbb{Q})$ and $\phi_L : I_2 \to \text{Gal}(L/\mathbb{Q})$. Show that $KL/K$ is unramified at 2 if and only if $\ker \phi_K \subset \ker \phi_L$.

(6) Using an argument like the one in class to construct $L/\mathbb{Q}$ with $LK/K$ unramified at finite primes, show that if $K$ is real quadratic and its discriminant has $k$ prime divisors, then the 2-rank of the narrow class group of $K$ is $k - 1$. (You can reference the argument in class and just explain what needs to be changed for this case.)

(7) Give a formula for $\# \text{Sur}(\mathbb{Z}_p^n, A)$ for a finite abelian $p$-group $A$ (in terms of any convenient numerical invariants of $|A|$).