MATH 273 FALL 2020: HOMEWORK 1

Due Thursday, Sep 17

(1) Given a field extension $K/\mathbb{Q}$ of degree n, how many subfields of $\bar{\mathbb{Q}}$ (bar denotes algebraic closure) are isomorphic to K?

(2) Given a field extension $K/\mathbb{F}_q(t)$ of degree n, how many subfields of $\overline{\mathbb{F}_q(t)}$ are isomorphic to K? (Hint: this is a "trick" question.)

(3) Give a graph G on n vertices, what is the probability that an Erdős–Rényi random graph is isomorphic to G? (Such a graph is obtained by taking n vertices and putting in each of the possible edges independently with probability 1/2.)

(4) Find a formula for $|\operatorname{Aut}(G)|$ for a finite abelian group G. (Hint: you may find that some ways of parametrizing the finite abelian groups lend themselves more easily than others to a nice looking formula.)

(5) Given that

$$\sum_{G \text{ fin. ab } p\text{-group}} \frac{1}{|\operatorname{Aut}(G)|} = \prod_{i \geq 1} (1 - p^{-i})^{-1},$$

if ν is the measure on finite abelian p-group proportional to $1/|\operatorname{Aut} G|$, find

$$\mathbb{P}_\nu(G \text{ is cyclic}).$$

(6) If $K/\mathbb{Q}$ is a Galois extension with $\operatorname{Gal}(K/\mathbb{Q}) \simeq \Gamma$, what finite $\mathbb{Z}[\Gamma]$-modules do you know cannot appear as Cl_K? (I expect you to find some that you can prove don't appear, but not to prove that any actually do appear.)

(7) Let $\mathcal{F}$ be some family of number fields, and μ_X be the uniform measure on those fields $K \in \mathcal{F}$ with $|\operatorname{Disc} K| \leq X$. Suppose for every finite abelian group A, we have that the limit

$$\lim_{X \to \infty} \mathbb{E}_{\mu_X}(\# \operatorname{Hom}(Cl_K, A)) \tag{1}$$

exists. Show that for every finite abelian group B, the limit

$$\lim_{X \to \infty} \mathbb{E}_{\mu_X}(\# \operatorname{Sur}(Cl_K, B)) \tag{2}$$

exists, and express the limits in (2) in terms of the limits in (1). (Sur denotes surjective homomorphisms.)