Due Thursday, Sep 17

(1) Given a field extension $K/\mathbb{Q}$ of degree $n$, how many subfields of $\overline{\mathbb{Q}}$ (bar denotes algebraic closure) are isomorphic to $K$?

(2) Given a field extension $K/\mathbb{F}_q(t)$ of degree $n$, how many subfields of $\overline{\mathbb{F}_q(t)}$ are isomorphic to $K$? (Hint: this is a “trick” question.)

(3) Give a graph $G$ on $n$ vertices, what is the probability that an Erdős–Rényi random graph is isomorphic to $G$? (Such a graph is obtained by taking $n$ vertices and putting in each of the possible edges independently with probability $1/2$.)

(4) Find a formula for $|\text{Aut}(G)|$ for a finite abelian group $G$. (Hint: you may find that some ways of parametrizing the finite abelian groups lend themselves more easily than others to a nice looking formula.)

(5) Given that

$$\sum_{G \text{ fin. ab } p\text{-group}} \frac{1}{|\text{Aut}(G)|} = \prod_{i \geq 1} (1 - p^{-i})^{-1},$$

if $\nu$ is the measure on finite abelian $p$-group proportional to $1/|\text{Aut} G|$, find $P_\nu(G \text{ is cyclic}).$

(6) If $K/\mathbb{Q}$ is a Galois extension with $\text{Gal}(K/\mathbb{Q}) \simeq \Gamma$, what finite $\mathbb{Z}[\Gamma]$-modules do you know cannot appear as $\text{Cl}_K$? (I expect you to find some that you can prove don’t appear, but not to prove that any actually do appear.)

(7) Let $\mathcal{F}$ be some family of number fields, and $\mu_X$ be the uniform measure on those fields $K \in \mathcal{F}$ with $|\text{Disc } K| \leq X$. Suppose for every finite abelian group $A$, we have that the limit

$$\lim_{X \to \infty} E_{\mu_X}(\# \text{Hom}(\text{Cl}_K, A))$$

exists. Show that for every finite abelian group $B$, the limit

$$\lim_{X \to \infty} E_{\mu_X}(\# \text{Sur}(\text{Cl}_K, B))$$

exists, and express the limits in (2) in terms of the limits in (1). (Sur denotes surjective homomorphisms.)